INTERMEDIATE MODULI SPACES OF STABLE MAPS

ANDREI MUSTAŢĂ AND MAGDALENA ANCA MUSTAŢĂ

Abstract. We describe the Chow ring with rational coefficients of $\overline{M}_{0,1}(\mathbb{P}^n, d)$ as the subring of invariants of a ring $B^*(\overline{M}_{0,1}(\mathbb{P}^n, d); \mathbb{Q})$, relative to the action of the group of symmetries S_d. We compute $B^*(\overline{M}_{0,1}(\mathbb{P}^n, d); \mathbb{Q})$ by following a sequence of intermediate spaces for $\overline{M}_{0,1}(\mathbb{P}^n, d)$.

Introduction

The moduli spaces of stable maps from curves to smooth projective varieties were introduced by M. Kontsevich and Y. Manin in [KM]. They provided the set-up for an axiomatic algebro-geometric approach to Gromov-Witten theory, generating beautiful results in enumerative geometry and mirror symmetry. Gromov-Witten invariants, defined as intersection numbers on the moduli spaces of stable maps, were computed by recurrence methods. An important role in these methods was played by the "boundary divisors" of the moduli space, parametrizing maps with reducible domains.

In the case when the domain curve is rational and the target is $\mathbb{P}^n$, the functor $\overline{\mathcal{M}}_{0,m}(\mathbb{P}^n, d)$ is represented by a smooth Deligne-Mumford stack, $\overline{M}_{0,m}(\mathbb{P}^n, d)$. Here the generic member is a smooth, degree d, rational curve in $\mathbb{P}^n$ with m distinct marked points. The boundary is made of degree d morphisms $\mu : \mathcal{C} \to \mathbb{P}^n$ from nodal m–pointed curves $\mathcal{C}$ of arithmetic genus 0, such that every contracted component of $\mathcal{C}$ has at least 3 special points: some of the m marked points or nodes.

The cohomology ring of $\overline{M}_{0,m}(\mathbb{P}^n, d)$ is not known in general. K.Behrend and A.O'Halloran in [BO] have outlined an approach for computing the cohomology ring for $m = 0$, relying on a method of Akildiz and Carell. They give a complete set of generators and relations for the case $d = 2$ and for the ring of $\overline{M}_{0,0}(\mathbb{P}^\infty, 3)$.

The main result of this paper is a description of the Chow ring with rational coefficients of $\overline{M}_{0,1}(\mathbb{P}^n, d)$. Our method is different from the one employed by K.Behrend and A.O'Halloran, relying on a sequence of intermediate moduli spaces. In Theorem 3.23, $A^*(\overline{M}_{0,1}(\mathbb{P}^n, d); \mathbb{Q})$ is expressed as the subring of invariants of a ring $B^*(\overline{M}_{0,1}(\mathbb{P}^n, d); \mathbb{Q})$, relative to the action of the group of symmetries S_d. We give a complete set of generators and

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relations for $B^*(\overline{M}_{0,1}(\mathbb{P}^n, d); \mathbb{Q})$, the geometric significance of which will be explained here in more detail.

Motivated by results in mirror symmetry, Givental in [G], and Lian, Liu and Yau in [LLY] have computed Gromov–Witten invariants for hypersurfaces in $\mathbb{P}^n$, via the Bott residue formula for a birational morphism

$$\Phi : \overline{M}_{0,0}((\mathbb{P}^n \times \mathbb{P}^1), (d, 1)) \to \mathbb{P}^n_d.$$

Here $\mathbb{P}^n_d := \mathbb{P}^{(n+1)(d+1)-1}$ parametrizes $(n+1)$ degree d– polynomials in one variable, modulo multiplication by constants. Following Givental, we will call the domain of Φ the graph space. We will use the short notation $G(\mathbb{P}^n, d)$ for it.

Of the various boundary divisors of $G(\mathbb{P}^n, d)$ and their images in $\mathbb{P}^n_d$, the most notable for us is $\overline{M}_{0,1}(\mathbb{P}^n, d) \times \mathbb{P}^1$, mapped by Φ into $\mathbb{P}^n \times \mathbb{P}^1$. The product $\overline{M}_{0,1}(\mathbb{P}^n, d) \times \mathbb{P}^1$ is embedded in $G(\mathbb{P}^n, d)$ as the space parametrizing split curves $\mathcal{C}_1 \cup \mathcal{C}_2$, were $\mathcal{C}_1$ comes with a degree $(d, 0)$ morphism to $\mathbb{P}^n \times \mathbb{P}^1$ and $\mathcal{C}_2$ comes with a degree $(0, 1)$ morphism. Our study starts from the diagram

$$\begin{tikzcd}
\overline{M}_{0,1}(\mathbb{P}^n, d) \times \mathbb{P}^1 \arrow[r] \arrow[d] & G(\mathbb{P}^n, d) \arrow[d, "\Phi"] \\
\mathbb{P}^n \times \mathbb{P}^1 \arrow[r] & \mathbb{P}^n_d.
\end{tikzcd}$$

A sequence of intermediate moduli spaces $G(\mathbb{P}^n, d, k)$ is constructed such that the morphism $\Phi : G(\mathbb{P}^n, d) =: G(\mathbb{P}^n, d, d) \to \mathbb{P}^n_d =: G(\mathbb{P}^n, d, 0)$ factors through $G(\mathbb{P}^n, d, k+1) \to G(\mathbb{P}^n, d, k)$. This also induces intermediate spaces $\overline{M}_{0,1}(\mathbb{P}^n, d, k)$. They are described via local $(S_d)^{n+1}$–covers $G(\mathbb{P}^n, d, k, \bar{t})$ for all homogeneous coordinate systems $\bar{t}$ on $\mathbb{P}^n$. These are modeled after the construction by W.Fulton and R.Pandharipande of the rigidified moduli spaces $\left(\overline{M}_{0,0}(\mathbb{P}^n, d, \bar{t})\right)_{\bar{t}}$, which form an étale cover for the stack $\overline{M}_{0,0}(\mathbb{P}^n, d)$.

There is a commutative diagram

$$\begin{tikzcd}
\overline{M}_{0,1}(\mathbb{P}^n, d, \bar{t}) \times \mathbb{P}^1 \arrow[r] \arrow[d] & G(\mathbb{P}^n, d, \bar{t}) \arrow[r] \arrow[d, "\Phi(t)"] & \mathbb{P}^1[(n+1)d] \arrow[d] \\
(\mathbb{P}^1)^n \times \mathbb{P}^1 \arrow[r] & \mathbb{P}^n_d(\bar{t}) \arrow[r] & (\mathbb{P}^1)^{(n+1)d}.
\end{tikzcd}$$

where $\mathbb{P}^1[(n+1)d]$ is the Fulton–MacPherson configuration space of $\mathbb{P}^1$ (see [FM]). $\mathbb{P}^n_d(\bar{t})$ is a torus bundle over $(\mathbb{P}^1)^{(n+1)d}$ and $G(\mathbb{P}^n, d, \bar{t})$ is its pullback to an open subset of the configuration space $\mathbb{P}^1[(n+1)d]$.

The morphism $\mathbb{P}^1[(n+1)d] \to (\mathbb{P}^1)^{(n+1)d}$ has been introduced and described as a sequence of blow–ups by W.Fulton and R.MacPherson in [FM]. We consider a different sequence of blow–ups, more symmetric with respect to the $(n+1)d$ points. If instead of $\mathbb{P}^1[(n+1)d]$ one considers $\overline{M}_{0,n}$, the resulting sequence is already known (see [Has], [T]). In our case we describe the intermediate spaces explicitly as moduli spaces, leading, via

pull–back of the torus bundles $\mathbb{P}^n_d(\bar{t})$, to $G(\mathbb{P}^n, d, k, \bar{t})$. Thus the morphism $G(\mathbb{P}^n, d, k+1, \bar{t}) \to G(\mathbb{P}^n, d, k, \bar{t})$ is simply a composition of regular blow–ups, and the blow–up loci are all smooth, equidimensional, transverse to each other and are mapped into each other by the natural $(S_d)^{n+1}$–action.

Following [V], there is a natural Deligne–Mumford stack associated to the coarse moduli scheme $G(\mathbb{P}^n, d, k)$. An étale cover of this stack is constructed out of quotients of the $\bar{t}$–covers by small groups. The above properties of the blow–up loci are preserved at the level of étale cover, but the blow–up is weighted. Therefore we will call the morphism $G(\mathbb{P}^n, d, k+1, \bar{t}) \to G(\mathbb{P}^n, d, k, \bar{t})$ a weighted blow–up along a regular local embedding.

In this paper we restricted our attention to an intersection–theoretical study of $\overline{M}_{0,1}(\mathbb{P}^n, d)$ and its intermediate spaces $\overline{M}_{0,1}(\mathbb{P}^n, d, k)$. The coarse moduli space and its associated Deligne–Mumford stack share the same Chow rings. The stack introduced above finely represents the moduli functor, which is shown in [MM].

The first intermediate space is a weighted projective fibration over $\mathbb{P}^n$, the class of its line bundle $\mathcal{O}(1)$ being the cotangent class ψ. The unique polynomial $P(t)$ having ψ as root in the cohomology of $\overline{M}_{0,1}(\mathbb{P}^n, d, 1)$ can be written in terms of the J–function of $\mathbb{P}^n$ as follows: $P(t) = t^{-1}J_d^{-1}$. These calculations are rooted in a simple form of Atyiah–Bott localization. Then the Chow rings of $\overline{M}_{0,1}(\mathbb{P}^n, d, k)$ and its substrata are computed by induction on k. In order to do this, an extended Chow ring is defined for a network generated by regular local embeddings, which morally mimics the Chow ring of the étale cover of the stack. There are no complete objects at the level of étale covers, but all potential generators of the Chow ring descend and glue to complete smooth Deligne–Mumford stacks $\overline{M}^k_I$. The extended Chow ring is generated by the classes of these stacks.

In our view $\overline{M}_{0,1}(\mathbb{P}^n, d)$ is the natural starting point in the study of the Chow ring of $\overline{M}_{0,m}(\mathbb{P}^n, d)$ for any m. On one hand, the Chow ring of $\overline{M}_{0,0}(\mathbb{P}^n, d)$ could be computed from that of $\overline{M}_{0,1}(\mathbb{P}^n, d)$ by analogy with the computation of the Chow ring of the Grassmannian Grass $(2, n)$ from that of the flag variety Flag $(1, 2, n)$, as we hope to show in an upcoming paper. On the other hand, the methods of this paper can be applied for the computation of the Chow ring of any $\overline{M}_{0,m}(\mathbb{P}^n, d)$ with $m > 1$ (see [MM]).

The plan of the paper is as follows: section 1 describes the intermediate spaces $G(\mathbb{P}^n, d, k)$, $\overline{M}_{0,1}(\mathbb{P}^n, d, k)$ and the morphisms among them. Section 2 contains a modular presentation of their canonical stratifications. Section 3 is an extended account of the induction steps involved in the computation of the Chow ring of $\overline{M}_{0,1}(\mathbb{P}^n, d)$, and the Appendix contains some calculations leading to the simplified final formula of the Chow ring.

This paper is based on an idea of the first author. The detailed construction of the intermediate moduli spaces and an outline of the method for computing the Chow ring of $\overline{M}_{0,1}(\mathbb{P}^n, d)$ are contained in Andrei Mustata's Ph.D thesis at the University of Utah. Subsequent work by the two authors

led to the present set–up for the construction of the ring $B^*(\overline{M}_{0,1}(\mathbb{P}^n, d); \mathbb{Q})$, and the computations that ensued.

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1. A description of the graph space

Enumerating rational curves in a projective variety V requires as starting point the existence of a compactification for the space of smooth rational curves in V. When a curve class $\beta \in H_2(V, \mathbb{Z})$ is fixed, Kontsevich–Manin's moduli space of stable maps $\overline{M}_{0,0}(V, \beta)$ provides such a suitable compactification. A *stable map* is a tuple (C, π, f) where $\pi : C \to S$ is a nodal curve over a scheme S, $f : C \to V$ is a map such that $f_*[C] = \beta$, and every geometric fiber of π over a point in S has only finitely many automorphisms that preserve the map f.

In particular, the *graph space* $G(\mathbb{P}^n, d) := \overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^1, (d, 1))$ was the background for the first results in Gromov–Witten theory (see [G], [LLY], [B]). On the other hand, the space of parametrized, degree d smooth rational curves in $\mathbb{P}^n$ admits another, naive compactification given by of $\mathbb{P}^n_d = \mathbb{P}^{(n+1)(d+1)-1}$, the space of $(n+1)$ degree (at most) d polynomials in one variable, modulo multiplication by a scalar. The existence of a natural birational morphism

$$\Phi : G(\mathbb{P}^n, d) \to \mathbb{P}^n_d$$

was instrumental in the computations mentioned above.

Our first observation is that Φ may be regarded as a natural transformation of moduli functors, as we shall discuss in Proposition 1.3. Given a stable map (C, π, f) with parametrization $\mu : C \to S \times \mathbb{P}^1$, the transformation $\Phi(S)$ contracts the unparametrized components of C. The morphism $f : C \to \mathbb{P}^n$ induces a rational map $f_0 : S \times \mathbb{P}^1 \to \mathbb{P}^n$. The points in the base locus of f_0 correspond to contracted components of C.

In the above context, an unparametrized component of a rational curve whose removal does not disconnect the curve is called a *tail.* One can conceive of various ways to factor the morphism Φ. For example, contraction of unparametrized components of C could proceed by sequentially contracting tails of increasing degree. In the following we define moduli problems for the intermediate steps we have just proposed:

Definition 1.1. Let k be a natural number, $0 \le k \le d$. Fix a small rational number ϵ such that $0 < \epsilon < 1$.

A k–stable family of degree d maps from parametrized rational curves into $\mathbb{P}^n$ is a tuple $(C, \mu, \mathcal{L}, e)$, where $\mu : C \to S \times \mathbb{P}^1$ is a morphism over S of degree 1 on each fiber C_s over $s \in S$, $\mathcal{L}$ is a line bundle on C, of degree d on each fiber C_s, and $e : \mathcal{O}^{n+1} \to \mathcal{L}$ is a morphism of fiber bundles such that:

(1) $(\mu^*\mathcal{O}_{S\times\mathbb{P}^1}(1) \otimes \omega_{C|S})^{d-k+\epsilon} \otimes \mathcal{L}$ is relatively ample over S,

(2) $\mathcal{G} := \operatorname{coker} e$, restricted to each fiber C_s, is a skyscraper sheaf, and
- $\dim \mathcal{G}_p \leq d-k$ for any $p \in C_s$, where $\mathcal{G}_p$ is the stalk of $\mathcal{G}$ at p;
- if $0 < \dim \mathcal{G}_p$ then $p \in C_s$ is a smooth point of C_s.

By condition (1), a k–stable map from C to $\mathbb{P}^n$ will have no tails of degree less than $d-k+1$, while by condition (2), points with multiplicity at most $d-k$ may be in the base locus of e. These points replace the smaller degree components of the usual stable maps. In the case $k=d$, the existence of ϵ makes formula (1) equivalent to the classical stability condition; in all other cases, ϵ may be taken to be zero.

As usual, a morphism between two families $(C, \mu, \mathcal{L}, e)$ and $(C', \mu', \mathcal{L}', e')$ of k–stable, degree d pointed maps consists of a Cartesian diagram

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "\mu"] & C' \arrow[d, "\mu'"] \\ S \arrow[r] & S' \end{tikzcd}$$

such that $g^*\mathcal{L}' \cong \mathcal{L}$ and e is induced by e' via the isomorphisms $\mathcal{O}_C^{n+1} \cong g^*\mathcal{O}_{C'}^{n+1}$ and $g^*\mathcal{L}' \cong \mathcal{L}$.

Notation . Let $G(\mathbb{P}^n, d, k)$ denote the functor of isomorphism classes of k–stable, degree d families of maps from parametrized rational curves to $\mathbb{P}^n$.

In the future, the same notation will be somewhat abusively employed for the corresponding coarse moduli scheme as well as for its moduli stack. Clearly, $G(\mathbb{P}^n, d, d) = G(\mathbb{P}^n, d)$ is the graph space, while $G(\mathbb{P}^n, d, 0)$ is $\mathbb{P}^n_d$.

A sequence of intermediate moduli functors is induced for the moduli space of stable maps with one marked point. The parametrized component is replaced here by the component containing the marked point:

Definition 1.2. Let k be a natural number, $0 < k \leq d$. Similarly we define a k–stable family of degree d pointed maps from parametrized rational curves into $\mathbb{P}^n$ by a tuple $(C, \pi, p_1, \mathcal{L}, e)$, where $\pi : C \to S$ is a flat family of genus zero nodal curves over S, $p_1 : S \to C$ is a section in the smooth locus of π, $\mathcal{L}$ is a line bundle of degree d on each fiber C_s, $s \in S$, and $e : \mathcal{O}^{n+1} \to \mathcal{L}$ is a morphism of fiber bundles satisfying condition (2) from above, such that:

(1) $\omega_{C|S}(p_1)^{d-k+\epsilon} \otimes \mathcal{L}$ is relatively ample over S,

and such that $\dim \mathcal{G}_{p_1(s)} = 0$ for any $s \in S$.

Notation . $\overline{M}_{0,1}(\mathbb{P}^n, d, k)$ will denote the functor of isomorphism classes of k–stable, degree d families of maps from one–pointed rational curves to $\mathbb{P}^n$. Here $\overline{M}_{0,1}(\mathbb{P}^n, d, d) = \overline{M}_{0,1}(\mathbb{P}^n, d)$.

Proposition 1.3. *There are natural transformations of functors*

$$G(\mathbb{P}^n, d, k+1) \to G(\mathbb{P}^n, d, k),$$

for any natural number k such that $0 \leq k < d$, and

$$\overline{M}_{0,1}(\mathbb{P}^n, d, k+1) \to \overline{M}_{0,1}(\mathbb{P}^n, d, k)$$

for k such that $0 < k < d$.

Proof. Let $(\pi : C_{k+1} \to S, \mu_{k+1}, \mathcal{L}_{k+1}, e_{k+1})$ be a $(k+1)$–stable family of degree d maps from parametrized rational curves into $\mathbb{P}^n$. Set

$$\mathcal{H} := (\mu_{k+1}^* \mathcal{O}_{S\times\mathbb{P}^1}(1) \otimes \omega_{C_{k+1}|S_{k+1}})^{d-k} \otimes \mathcal{L}_{k+1}.$$

For sufficiently large N, we consider $C_k = \mathrm{Proj}(\oplus_{m\geq 0}\pi_*\mathcal{H}^{mN})$. The sections of $\mathcal{H}^N$ induce a morphism $f_k : C_{k+1} \to C_k$, which collapses the locus along which $\mathcal{H}$ fails to be relatively ample, and maps its complement birationally onto its image. Note that $\mathcal{H}$ fails to be ample precisely along tails of degree $d-k$. Let $\mathcal{O}_{C_{k+1}}(D)$ denote the line bundle associated to that locus. (This is the pullback of a line bundle on the universal family on the smooth Deligne–Mumford stack $G(\mathbb{P}^n, d, k+1)$ introduced in Proposition 1.11). Then $\mathcal{L}_{k+1}\otimes \mathcal{O}_{C_{k+1}}((d-k)D)$ descends to a line bundle $\mathcal{L}_k$ on C_k, with $(n+1)$ global sections determined by the composition:

$$f_k^*\mathcal{O}_{C_k}^{n+1} \to \mathcal{O}_{C_{k+1}}^{n+1} \to \mathcal{L}_{k+1} \to \mathcal{L}_{k+1} \otimes \mathcal{O}_{C_{k+1}}((d-k)D).$$

For any geometric point $s \in S$ such that the fiber C_{k+1s} decomposes as $F_1 \cup F_2$ with $F_1 \cap F_2 = \{q\}$, and such that f_k collapses F_2 and is an isomorphism on F_1, the following relations over F_1:

$$\omega_{C_{k+1s}|F_1} = \omega_{C_{ks}|F_1} \otimes \mathcal{O}_{F_1}(q),$$

$$\mathcal{G}_{k+1|F_1} = \mathcal{G}_{k|F_1} \oplus k_q^{d-k},$$

insure that conditions (1) and (2) in Definition 1.1 are satisfied.

□

Turning our attention to the representability of the functors defined above, we recall the notion of rigid stable maps, which makes the local structure of the moduli spaces accessible via étale atlases.

In the case of $G(\mathbb{P}^n, d)$, the rigid structure accurately transcribes the construction of [FP] (Proposition 3), with the only modification that non–parametrized rational curves are replaced by parametrized ones. Here are the main points:

Let $\bar{t} = (t_0 : ... : t_n)$ stand for a choice of a homogeneous coordinate system for $\mathbb{P}^n$.

Definition 1.4. A $(\bar{t})$–rigid family of degree $(d,1)$ stable maps from genus 0 curves to $\mathbb{P}^n \times \mathbb{P}^1$ consists of data $(C, \pi, \mu, \{q_{i,j}\}_{0\leq i\leq n, 1\leq j\leq d})$ where

1. $(\pi\colon \mathcal{C} \to S, \mu\colon \mathcal{C} \to \mathbb{P}^n \times \mathbb{P}^1)$ is a family of stable, degree (d,1) maps from genus 0 curves to $\mathbb{P}^n \times \mathbb{P}^1$.
2. $(\mathcal{C} \to S\times\mathbb{P}^1, \{q_{i,j}\}_{0\leq i\leq n, 1\leq j\leq d})$ is a family of $(n+1)d$ –pointed, genus zero, stable parametrized curves.

(3) For any integer i such that $0 \leq i \leq n$, the following equality of Cartier divisors holds:

$$\mu^*(t_i) = \sum_{j=1}^{d} q_{i,j}$$

Notation . The contravariant functor of isomorphism classes of $(\bar{t})$–rigid families of degree $(d,1)$ stable maps from rational curves to $\mathbb{P}^n \times \mathbb{P}^1$ is denoted by $G(\mathbb{P}^n, d, \bar{t})$.

Lemma 1.5. *$G(\mathbb{P}^n, d, \bar{t})$ is finely represented by the total space of a torus bundle over a Zariski open subset of $\overline{M}_{0,(n+1)d}(\mathbb{P}^1, 1)$.*

The fine moduli scheme will also be denoted by $G(\mathbb{P}^n, d, \bar{t})$. The proof does not differ in any essential way from that of [FP], Proposition 3. We translate the main steps of the construction to our setup:

$\overline{M}_{0,(n+1)d}(\mathbb{P}^1, 1)$ is the Kontsevich–Manin moduli space of $(n+1)d$–pointed, degree 1 stable maps and $f : \overline{M}_{0,(n+1)d+1}(\mathbb{P}^1, 1) \to \overline{M}_{0,(n+1)d}(\mathbb{P}^1, 1)$ is its universal family, with $(n+1)d$ sections $\{q_{i,j}\}_{0 \leq n, 1 \leq d}$.

Consider the line bundles $\mathcal{H}_i = \mathcal{O}_{\overline{M}_{0,(n+1)d}(\mathbb{P}^1,1)}(\sum_{j=1}^{d} q_{i,j})$, for $0 \leq i \leq n$. Let $j : B \hookrightarrow \overline{M}_{0,(n+1)d}(\mathbb{P}^1, 1)$ be the open subscheme parametrizing degree 1 pointed maps $C \to \mathbb{P}^1$ such that, for every fixed component C' of C, the subsets of the marked point sets $\{q_{i,j}\}_j$ lying on C' have the same cardinal for all $i \in \{0, ..., n\}$. Equivalently, $\mathcal{G}_i := \pi_{B*}\bar{j}^*(\mathcal{H}_0^{-1} \otimes \mathcal{H}_i)$ is locally free, and the canonical map $\pi_B^* \pi_{B*}\bar{j}^*(\mathcal{H}_0^{-1} \otimes \mathcal{H}_i) \to \bar{j}^*(\mathcal{H}_0^{-1} \otimes \mathcal{H}_i)$ is an isomorphism for every $1 \leq i \leq n$, where $\bar{j}, \pi_B$ are the canonical projections from $B \times_{\overline{M}_{0,(n+1)d}(\mathbb{P}^1,1)} \overline{M}_{0,(n+1)d+1}(\mathbb{P}^1, 1)$ to $\overline{M}_{0,(n+1)d+1}(\mathbb{P}^1, 1)$ and B, respectively. We say that the map j is balanced. Moreover, every balanced morphism $X \to \overline{M}_{0,(n+1)d}(\mathbb{P}^1, 1)$ factors through j. Let then $\tau_i : Y_i \to B$ denote the total space of the canonical $\mathbb{C}^*$ –bundle associated to $\mathcal{G}_i$. The torus bundle $Y := Y_1 \times_B Y_2 \times_B \ldots \times_B Y_n$ represents the functor $G(\mathbb{P}^n, d, \bar{t})$.

The $(n+1)$–th power $G := (S_d)^{n+1}$ of the group of permutations S_d acts on $G(\mathbb{P}^n, d, \bar{t})$ as permutations of the marked points $\{q_{i,j}\}$. The coarse moduli scheme $G(\mathbb{P}^n, d)$ is obtained by gluing quotients of $G(\mathbb{P}^n, d, \bar{t})$ by G, for various homogeneous coordinate systems $\bar{t}$. Concomitantly, $\bigsqcup_{\bar{t}} G(\mathbb{P}^n, d, \bar{t})$ is an étale atlas of the stack $G(\mathbb{P}^n, d)$.

There is also an appropriate notion of $\bar{t}$–rigidity for k–stable maps, leading to the representability of $G(\mathbb{P}^n, d, k)$. The case of $\mathbb{P}^n_d(\bar{t}) = G(\mathbb{P}^n, d, 0, \bar{t})$ is most at hand. The data $(S \times \mathbb{P}^1, \mu, \{q_{i,j}\}_{0 \leq i \leq n, 1 \leq j \leq d})$ make a $\bar{t}$–rigid, 0–stable family of maps if conditions (1) and (3) in Definition 1.4 are satisfied, with 0–stability replacing the usual stability in (1). No incidence conditions need be placed on $\{q_{i,j}\}_{i,j}$. There is a natural, rank $(n+1)$ torus bundle over $(\mathbb{P}^1)^{(n+1)d}$ representing $\mathbb{P}^n_d(\bar{t})$, the functor of isomorphism families of

$\bar{t}$–rigid, 0–stable maps. Indeed, an algebraic map $\mathbb{P}^1 \to \mathbb{P}^n$ is specified by $(n+1)$ independent hyperplane sections and $(n+1)$ constants, the hyperplane sections providing the roots of $(n+1)$ polynomials, and the $(n+1)$ constants acting as coefficients. For a more rigorous construction, set $P := (\mathbb{P}^1)^{(n+1)d}$ and for all pairs (i,j) such that $0 \leq i \leq n$, $1 \leq j \leq d$, let $\pi_{i,j} : P \to \mathbb{P}^1$ be the (i,j)–th projection. Define

$$\mathcal{F}_i := \pi_{i,1}^*(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes \pi_{i,2}^*(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes \cdots \otimes \pi_{i,d}^*(\mathcal{O}_{\mathbb{P}^1}(1))$$

and let F_i be the total space of the canonical $\mathbb{C}^*$–bundle associated to $\mathcal{F}_i \otimes \mathcal{F}_0^{-1}$. Then a standard translation of the arguments in Proposition 3 of [FP], $\mathbb{P}^n_d(\bar{t})$ is finely represented by $F := F_1 \times_P F_2 \times_P ... \times_P F_n$. Notice that if $\pi_P : P \times \mathbb{P}^1 \to P$ is the "universal family" over P, and $\{s_{i,j}\}_{i,j}$ are its canonical sections, then $\mathcal{O}_{P\times\mathbb{P}^1}(s_{i,j}) = \pi_P^*\mathcal{F}_i \otimes \pi_1^*\mathcal{O}_{\mathbb{P}^1}(d)$, hence the analogy with [FP].

Lemma 1.6. *The group $G = (S_d)^{(n+1)}$ acts on the universal family of $\mathbb{P}^n_d(\bar{t})$ by permutations of the sections $\{q_{i,1}, q_{i,2}, ..., q_{i,d}\}$, for each $0 \leq i \leq n$. Consider the induced action on $\mathbb{P}^n_d(\bar{t})$. The GIT quotients $\mathbb{P}^n_d(\bar{t})/G$ for various $\bar{t}$–s glue together to $\mathbb{P}^n_d$, which finely represents the functor $G(\mathbb{P}^n, d, 0)$.*

Proof. By the reasons spelled out in Proposition 4 of [FP], the quotients $\mathbb{P}^n_d(\bar{t})/G$ for various $\bar{t}$ do patch together to a coarse moduli scheme. Moreover, this is a smooth scheme, $\mathbb{P}^n_d(\bar{t})/G$ being a torus bundle over $(\mathbb{P}^d)^{n+1}$. The essential difference here is the existence of a universal family over the scheme $\mathbb{P}^n_d$: a canonical rational map $\mu : \mathbb{P}^n_d \times \mathbb{P}^1 \to \mathbb{P}^n$. Given any point $(p, x) \in \mathbb{P}^n_d \times \mathbb{P}^1$, this map evaluates each of the $(n+1)$ degree d polynomials corresponding to p at the point $x \in \mathbb{P}^1$. For a fixed coordinate system $\bar{t} = (t_0 : ... : t_{n+1})$, let $U_{\bar{t}} \subset \mathbb{P}^n_d$ be the open subset made of points p such that $\mu(\{p\} \times \mathbb{P}^1) \not\subset \bigcup_i (t_i = 0)$. There is an obvious bijection $U_{\bar{t}} \to \mathbb{P}^n_d(\bar{t})/G$ and, since both spaces are smooth, an isomorphism. □

Thus both the graph space $G(\mathbb{P}^n, d)$ and $\mathbb{P}^n_d$ admit local finite covers $G(\mathbb{P}^n, d, \bar{t})$ and $\mathbb{P}^n_d(\bar{t})$, respectively. There is a birational morphism $\Psi : \overline{M}_{0,(n+1)d}(\mathbb{P}^1, 1) \to (\mathbb{P}^1)^{(n+1)d}$, the product of all the evaluation maps. Due to the canonical nature of the previous constructions, the torus bundle $G(\mathbb{P}^n, d, \bar{t})$ over $B \subset \overline{M}_{0,(n+1)d}(\mathbb{P}^1, 1)$, is the pullback of $\mathbb{P}^n_d(\bar{t})$ via $\Psi_{|B}$.

We will now see how the schemes $G(\mathbb{P}^n, d, k)$ may be obtained by gluing quotients by $(S_d)^{n+1}$ of torus bundles $G(\mathbb{P}^n, d, k, \bar{t})$ over some bases. To find appropriate bases for these bundles, one needs to factor the morphism Ψ into intermediate steps:

Set $N := (n+1)d$. The space $\overline{M}_{0,N}(\mathbb{P}^1, 1)$ is also known as the Fulton–MacPherson compactified configuration space of N distinct points in $\mathbb{P}^1$, namely the closure (through the diagonal embedding) of $(\mathbb{P}^1)^N \setminus \bigcup_{i,j} \Delta_{i,j}$ in $(\mathbb{P}^1)^N \times \prod_{|S|\geq 2} Bl_{\Delta^S}(\mathbb{P}^1)^S$, where $\Delta_{i,j}$ are the large diagonals in $(\mathbb{P}^1)^N$, and

for all subsets $S \subset \{1, ..., N\}$, $Bl_{\Delta^S}(\mathbb{P}^1)^S$ denotes the blow–up of the corresponding Cartesian product along its small diagonal. From now on this compactification will be denoted by $\mathbb{P}^1[N]$. Thus $\mathbb{P}^1[N]$ may be constructed from $(\mathbb{P}^1)^N$ by a sequence of blow–ups of (strict transforms of) all diagonals in $(\mathbb{P}^1)^N$. A different order of blow–ups than the one considered by W.Fulton and R.MacPherson in [FM] also leads to the same result, while the intermediate steps are exactly the bases of $\{G(\mathbb{P}^n, d, k, \bar{t})\}_k$ previously sought:

Notation . Let K be an integer such that $0 \leq K \leq N$. $\mathbb{P}^1[N, K]$ will denote the closure through the diagonal embedding of $(\mathbb{P}^1)^N \setminus \bigcup_{i,j} \Delta_{i,j}$ in $(\mathbb{P}^1)^N \times \prod_{|S|>N-K} Bl_{\Delta^S}(\mathbb{P}^1)^S$, where S stands for subsets $S \subset \{1, ..., N\}$.

The point here is that for all subsets S such that $|S| = N - K$, all the strict transforms $\tilde{\Delta}^S$ in $\mathbb{P}^1[N, K]$ intersect transversely, and $\mathbb{P}^1[N, K + 1]$ is obtained by blowing up $\mathbb{P}^1[N, K]$ along these transforms, in any order. Moreover, at each step, the scheme $\mathbb{P}^1[N, K]$ finely represents a moduli problem of stable parametrized pointed rational curves:

Definition 1.7. Consider a morphism $\phi : C \to S \times \mathbb{P}^1$ of degree 1 over each geometric fiber C_s with $s \in S$, and N marked sections of $C \to S$. The morphism will be called K–stable if the following conditions are satisfied for all $s \in S$:

i) not more than $N - K$ of the marked points in C_s coincide;

ii) any ending irreducible curve in C_s, except the parametrized one, contains more than $N - K$ marked points (here an ending curve is a curve such that, if removed from C, the remaining curve is connected);

iii) all the marked points are smooth points of the curve C_s and C_s has finitely many automorphisms preserving the marked points and the map to $\mathbb{P}^1$.

A similar definition of K–stability can be given for unparametrized rational pointed curves, but then one needs to choose a special point among the marked points; let it be denoted by P_1. This marked point is not allowed to coincide with any other marked point. In condition ii), the parametrized component of C is replaced by the component containing P_1.

Proposition 1.8. *The smooth scheme $\mathbb{P}^1[N, K]$ finely represents the functor of isomorphism families of K–stable parametrized rational curves.*

Proof. The proposition may be checked by increasing induction on K. Given the universal family $U[N, K] \to \mathbb{P}^1[N, K]$ with sections $\{s_i\}_{1 \leq i \leq N}$, then the universal family $U[N, K + 1]$ is constructed by blowing up $\mathbb{P}^1[N, K + 1] \times_{\mathbb{P}^1[N,K]} U[N, K]$ along the loci where $N - K$ of the strict transforms $\tilde{s}_i : \mathbb{P}^1[N, K + 1] \to \mathbb{P}^1[N, K + 1] \times_{\mathbb{P}^1[N,K]} U[N, K]$ intersect. The transversality properties of these loci lead to the stability conditions in Definition 1.7.

Given any $(K + 1)$–stable family of parametrized rational curves $C \to S \times \mathbb{P}^1$, one may contract the tails with only $N - K$ marked points and obtain

a K–stable parametrized curve C' over S. The fact that the morphisms $S \to \mathbb{P}^1[N,k]$ and $C' \to U[N,K]$ canonically lift to $S \to \mathbb{P}^1[N,K+1]$ and $C \to U[N,K+1]$ results from the universality property of the blow–up. $\square$

For example, $\mathbb{P}^1[N,0] = (\mathbb{P}^1)^N$ and $\mathbb{P}^1[N,N] = \mathbb{P}^1[N]$.

Let $\bar{t} = (t_0 : \ldots : t_n)$ stand for a choice of a homogeneous coordinate system for $\mathbb{P}^n$. We are now ready to define the notion of $\bar{t}$–rigid, k–stable family of maps by relaxing conditions (1) and (2) in Definition 1.4:

Definition 1.9. A k–stable family of degree d, $\bar{t}$–rigid maps from rational parametrized curves into $\mathbb{P}^n$ is a tuple $(C, \mu, \mathcal{L}, e, \{q_{i,j}\}_{0\leq i\leq n, 1\leq j\leq d})$, such that:

1. $(C, \mu, \mathcal{L}, e)$ is a k–stable family of degree d maps from parametrized rational curves into $\mathbb{P}^n$;
2. $(C, \mu, \{q_{i,j}\}_{0\leq i\leq n, 1\leq j\leq d})$ is an $(n+1)k$ – stable family of parametrized curves;
3. via the natural isomorphism $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \cong H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}^{n+1})$,
$$(e(\bar{t}_i)) = \sum_{j=1}^{d} q_{i,j}.$$

Notation . The contravariant functor of isomorphism classes of $(\bar{t})$–rigid families of degree $(d,1)$, k–stable maps from rational curves to $\mathbb{P}^n \times \mathbb{P}^1$ is denoted by $G(\mathbb{P}^n, d, k, \bar{t})$.

Lemma 1.10. *$G(\mathbb{P}^n, d, k, \bar{t})$ is finely represented by the total space of a torus bundle over a Zariski open subset of $\mathbb{P}^1[(n+1)d, (n+1)k]$.*

The fine moduli scheme will also be denoted by $G(\mathbb{P}^n, d, k, \bar{t})$. The basis of the torus bundle is the open subset $U_k \subset \mathbb{P}^1[(n+1)d, (n+1)k]$ parametrizing tuples $(C, \{q_{i,j}\}_{0\leq i\leq n, 1\leq j\leq d})$, such that the line bundle $\mathcal{O}_C(\sum_{j=1}^{d} q_{i,j})$ does not depend on i. $G(\mathbb{P}^n, d, k, \bar{t})$ is pullback of $\mathbb{P}^n_d(\bar{t})$ to U_k.

We have discussed how $\overline{M}_{0,1}(\mathbb{P}^n, d)$ is embedded into $G(\mathbb{P}^n, d)$. A rational pointed unparametrized curve, attached to a parametrized $\mathbb{P}^1$ by identifying the marked point with $0 \in \mathbb{P}^1$ becomes parametrized. Accordingly, the spaces $\overline{M}_{0,1}(\mathbb{P}^n, d, k)$ introduced in Definition 1.2 also admit local finite $(S_d)^{n+1}$–covers $\overline{M}_{0,1}(\mathbb{P}^n, d, k, \bar{t})$, which are torus bundles over some bases. $\overline{M}_{0,1}(\mathbb{P}^n, d, \bar{t})$ for instance, has an open subset in $\overline{M}_{0,(n+1)d+1} \hookrightarrow \overline{M}_{0,(n+1)d}(\mathbb{P}^1, 1)$ as basis, and $\overline{M}_{0,1}(\mathbb{P}^n, d, 1, \bar{t})$ stands over $\mathbb{P}^{(n+1)d-2}$, a fiber in the exceptional divisor of $\mathbb{P}^1[(n+1)d, 1] = \mathrm{Bl}_{\Delta}(\mathbb{P}^1)^{(n+1)d}$. Definition 1.9 may be adapted to $\overline{M}_{0,(n+1)d+1}(k)$, by replacing the parametrized component with the one containing the first marked point, and asking that this point is always distinct from the others. In [K] Kapranov has already described the construction of $\overline{M}_{0,N+1}$ by successive blow–ups of $\mathbb{P}^{N-2}$, while Thaddeus remarked in [T] that the order of blow–ups is not important. The

intermediate steps employed by us are a particular case of Hassett's weighted pointed stable curves. ([Has]).

Proposition 1.11. *The coarse moduli schemes $G(\mathbb{P}^n, d, k)$ are obtained by gluing $(S_d)^{n+1}$–quotients of the rigidified moduli spaces $G(\mathbb{P}^n, d, k, \bar{t})$, for all homogeneous coordinate systems $\bar{t}$ on $\mathbb{P}^n$. The same is true for $\overline{M}_{0,1}(\mathbb{P}^n, d, k)$. There are natural structures of smooth Deligne–Mumford stacks associated to both $G(\mathbb{P}^n, d, k)$ and $\overline{M}_{0,1}(\mathbb{P}^n, d, k)$.*

Proof. We have discussed how the schemes $G(\mathbb{P}^n, d, k)$ and $\overline{M}_{0,1}(\mathbb{P}^n, d, k)$ are locally quotients of smooth varieties by the finite group $(S_d)^{n+1}$. Consequently, they are the moduli spaces of some smooth Deligne–Mumford stacks (Proposition 2.8 from [V]). The same notations will be used for these stacks as for their schemes. Chevalley–Shepard–Todd in [ST] show that the quotient of a smooth variety by a small group (a group generated by elements that are invariant along some divisors) is a smooth variety. Thus for general k, an étale cover of the stack $G(\mathbb{P}^n, d, k)$ will be obtained by factoring an appropriate neighborhood of any point $x \in G(\mathbb{P}^n, d, k, \bar{t})$ by the largest small normal subgroup H_x of the stabilizer $Stab_x \subset (S_d)^{n+1}$. This is exactly the subgroup whose elements fix the points of the universal curve over x, and thus fail to contribute to the stacky structure of the space of non–rigid k–stable maps. A detailed proof that $G(\mathbb{P}^n, d, k)$ and $\overline{M}_{0,1}(\mathbb{P}^n, d, k)$ are the stacks of k– stable maps is written in [MM].

When $k = d$, there are no small subgroups in $Stab_x$, and the various $G(\mathbb{P}^n, d, \bar{t})$ form an étale cover of the stack $G(\mathbb{P}^n, d)$. For $\mathbb{P}^n_d(\bar{t})$, the entire group $(S_d)^{n+1}$ is small and $\mathbb{P}^n_d$ is obviously smooth. For k in the middle, the map from $G(\mathbb{P}^n, d, k, \bar{t})$ to $G(\mathbb{P}^n, d, k)$ factors as a composition of a GIT quotient with an étale map of stacks. □

2. Substrata of intermediate spaces

Notation . From now on, the moduli space $\overline{M}_{0,1}(\mathbb{P}^n, d, k)$ will be denoted simply by $\overline{M}^k$.

There is a canonical stratification of $\overline{M}_{0,1}(\mathbb{P}^n, d)$ (and similarly of the graph space $G(\mathbb{P}^n, d)$), corresponding to splitting types of curves. The images in $\overline{M}^k$ induce a canonical stratification. We index the strata by nested subsets I of the power set of $D := \{1, ..., d\}$. Elements of I label marked points or subcurves of the generic curve parametrized by the stratum. The incidence relations among elements of I reflect incidence relations among the assigned marked points and subcurves.

Notation . Let $\mathcal{P}$ denote the power set of D, and let

$$\mathcal{P}_k = \{h \in \mathcal{P}; |h| = k\}$$

and similarly let $\mathcal{P}_{>k}$ be the set of cardinal $> k$ subsets of D, $\mathcal{P}_{\leq k}$, the set of cardinal $\leq k$ subsets. Similarly, for $I \subset \mathcal{P}$, set $I_k := I \bigcap \mathcal{P}_k$ etc.

Definition 2.1. $I \subset \mathcal{P} \setminus \{\emptyset, D\}$ is a nested set if, for any two $h, h' \in I$, the intersection $h \cap h'$ is either h, h' or $\emptyset$.

Definition 2.2. Fix a positive number $k < d$ and a nested set $I \subset \mathcal{P} \setminus \{\emptyset, D\}$ such that $h \cap h' = \emptyset$ for any distinct $h, h' \in I_{\leq d-k}$. An I–type, k–stable, degree d map from a rational curve into $\mathbb{P}^n$ is a tuple

$$(C, p_1, \{p_h\}_{h \in I_{\leq d-k}}, \{C_h\}_{h \in I_{>d-k}}, \mathcal{L}, e)$$

of a k–stable, degree d pointed map $(C, p_1, \mathcal{L}, e)$, together with marked points $\{p_h\}_{h \in I_{\leq d-k}}$ and connected subcurves $\{C_h\}_{h \in I_{>d-k}}$ satisfying the following properties:

(1) $\forall h \in I_{>d-k}$, $p_1 \notin C_h \subset C$ and $\deg \mathcal{L}_{|C_h} = |h|$;
(2) $\forall h \in I_{\leq d-k}$, $\dim \operatorname{coker} e_{p_h} = |h|$;
(3) compatibility of incidence relations:
  - $\forall h \in I_{\leq d-k}$, $\forall h' \in I_{>d-k}$, $h \subset h'$ iff $p_h \in C_{h'}$;
  - $\forall h, h' \in I_{>d-k}$, if $h' \subset h$, then $C_{h'} \subset C_h$, if $h \subset h'$, then $C_h \subset C_{h'}$, otherwise $C_h \cap C_{h'} = \emptyset$;

A curve C which admits a set of points and components with the above properties is said to be of I–splitting type.

Intuitively, the points in h may be thought of as the pullback on C_h of a hyperplane divisor in $\mathbb{P}^n$.

Notation . Given any nested set $I \subset \mathcal{P}$, let $G_I \subset S_d$ be the largest subgroup that keeps each $h \in I$ fixed. G_I decomposes into a direct sum of permutation groups $S_{h'}$, where $h' \in \mathcal{P}$ is one of the sets $h \setminus (\bigcap_{h'' \in I, h'' \neq h} h'')$ for $h \in I$, or $D \setminus (\bigcap_{h'' \in I} h'')$.

Proposition 2.3. *a) The functor of isomorphism families of type I, k–stable, degree d maps into $\mathbb{P}^n$ is coarsely represented by a scheme $\overline{M}_I^k$, having a canonically associated smooth Deligne–Mumford stack $\overline{\mathbf{M}}_I^k$. (This stack actually finely represents the functor).*

b) Given two nested sets $I \subset J$ as above, then $\bar{\phi}_J^I : \overline{\mathbf{M}}_J^k \to \overline{\mathbf{M}}_I^k$ is a regular local embedding of stacks, as long as $(J \setminus I)_{<d-k} = \emptyset$. In particular, $\overline{\mathbf{M}}_I^k$ admits a regular local embedding into $\overline{\mathbf{M}}^k := \overline{\mathbf{M}}_{0,1}(\mathbb{P}^n, d, k)$. The image is a closed substratum $\overline{\mathbf{M}}^k(I)$ of $\overline{\mathbf{M}}_{0,1}(\mathbb{P}^n, d, k)$ parametrizing k–stable, degree d–maps from curves of I–splitting type into $\mathbb{P}^n$.

c) If $|I \cap \mathcal{P}_l| = m > 1$ for some positive l, then the group of symmetries S_m acts nontrivially on the stack $\overline{\mathbf{M}}_I^k$, permuting the elements of I_l, and for any $h, h' \in I_l$, the transposition τ of $\{h, h'\}$ induces a commutative diagram

$$\begin{tikzcd}
\overline{\mathbf{M}}_I^k \arrow[r, "\tau"] \arrow[d] & \overline{\mathbf{M}}_I^k \arrow[d] \\
\overline{\mathbf{M}}_{I \setminus \{h\}}^k \arrow[r, "="] & \overline{\mathbf{M}}_{I \setminus \{h'\}}^k
\end{tikzcd}$$

Proof. a) A finite presentation for $\overline{M}^k_I$ is constructed via rigid k–stable maps. We fix a homogeneous coordinate system $\bar{t}$ on $\mathbb{P}^n$. Recall the sequence of morphisms:

$$\overline{M}_{0,1}(\mathbb{P}^n, d, k, \bar{t}) \hookrightarrow G(\mathbb{P}^n, d, k, \bar{t}) \xrightarrow{f_{\bar{t}}} \mathbb{P}^n_d(\bar{t}) \xrightarrow{p_{\bar{t}}} (\mathbb{P}^1)^{d(n+1)},$$

where $f_{\bar{t}}$ is the composition of blow–down morphisms described in section 1 and $\mathbb{P}^n_d(\bar{t})$ is a $(\mathbb{C}^*)^n$–torus over an open subset of $(\mathbb{P}^1)^{d(n+1)}$.

Let $N = \{0, ..., n\}$ and let $\Delta_{N\times h}$ denote the diagonal in $(\mathbb{P}^1)^{d(n+1)}$ where the coordinates corresponding to the set $N \times h \subset N \times \{1, ..., d\}$ agree. $\Delta^k_{N\times h}(\bar{t}) \subset G(\mathbb{P}^n, d, k, \bar{t})$ is defined inductively as follows: $\Delta^0_{N\times h}(\bar{t}) = p^{-1}_{\bar{t}}(\Delta_{N\times h})$, and $\Delta^k_{N\times h}(\bar{t})$ is the strict transform of $\Delta^{k-1}_{N\times h}(\bar{t})$ by the k–th blow–up, except at the $|h|$–th step when the total transform is considered instead. Define $D^k_{N\times h}(\bar{t}) := \Delta^k_{N\times h}(\bar{t}) \cap \overline{M}_{0,1}(\mathbb{P}^n, d, k, \bar{t})$ and $D^k_{N\times I}(\bar{t}) := \bigcap_{h\in I} D^k_{N\times h}(\bar{t})$. $D^k_{N\times I}(\bar{t})$ parametrizes I–type, k–stable, degree d, $\bar{t}$–rigid maps, in the sense of Definitions 1.9 and 2.2. We will also denote this space by $\overline{M}^k_I(\bar{t})$. By the standard method employed in Proposition 4 of [FP], the GIT quotients

$$\left(\coprod_{[g]\in(S_d/G_I)^{n+1}} D^k_{g(N\times I)}(\bar{t}) \right) /(S_d)^{n+1} \cong D^k_{N\times I}(\bar{t})/(G_I)^{n+1}$$

glue together to form the coarse moduli space $\overline{M}^k_I$. The étale presentation of the stack $\overline{M}^k_I$ is formed, as in Proposition 1.11, from GIT quotients of small neighborhoods of points $x \in \overline{M}^k_I(\bar{t})$ by the largest small subgroup in the stabilizer of x in $(G_I)^{n+1}$, and for various $\bar{t}$.

b) If I, J are nested sets as above, and if $I \subset J$, then there is an obvious morphism $\bar{\phi}^I_J : \overline{M}^k_J \to \overline{M}^k_I$, induced by inclusion on the $\bar{t}$–covers, as $G_J \subset G_I$. Moreover, if $(J \setminus I)_{<d-k} = \emptyset$, then for any $x \in \overline{M}^k_J(\bar{t}) \hookrightarrow \overline{M}^k_I(\bar{t})$, the largest small subgroup of its stabilizer in $(G_J)^{n+1}$ coincides with that in $(G_I)^{n+1}$. By this feature the morphism $\bar{\phi}^I_J$ is a regular local embedding, as seen on the étale covers of the stacks. This is not the case if there is an element $h \in (J \setminus I)_{<d-k}$: then for most $h'' \in \mathcal{P}_{\leq d-k}$ such that $h \subset h''$, and for $x \in D^k_{N\times h''}(\bar{t}) \cap D^k_{N\times J}(\bar{t})$, the small group $(S_{h''})^{n+1}$ is a summand in the stabilizer of x in $(G_I)^{n+1}$, while $G_J \cap S_{h''} = S_h \oplus S_{h''\setminus h}$. This prevents the morphism $\bar{\phi}^I_J$ from being a regular local embedding in this case.

c) follows from the local structure described in a). □

Notation . Let $I \subset \mathcal{P}$ be a nested set. Then the largest finite covers of $\overline{M}^k_I$ which embedd into $\overline{M}_{0,1}(\mathbb{P}^n, d, k, \bar{t})$ are $\coprod_{[g']\in(S_d/G_I)^n} D^k_{(g_0,g')(N\times I)}(\bar{t})$, for $g_0 \in S_d/G_I$. This follows from the stability condition (2) in Definition 1.9. For a point x of this cover and $t := (x, \bar{t})$, let $\overline{M}^k_I(t)$ denote the étale local cover of $\overline{M}^k_I$ obtained as a quotient by the appropriate small group of a neighborhood of x in the above finite cover, as in Proposition 1.11. The

étale covers $\{\overline{M}^k_I(t)\}_t$ organize the stacks $\{\overline{M}^k_I\}_I$ into an S_d– network of regular local embeddings, which shall be discussed in detail in section 3.

Consider a nested subset $I \subset \mathcal{P}$. At the level of local $(G_I)^{n+1}$– covers, the morphism $f^k_I(\bar{t}) : \overline{M}^{k+1}_I(\bar{t}) \to \overline{M}^k_I(\bar{t})$ factors into a sequence of regular blow–ups and projections of projective bundles. The blow–ups are along $\overline{M}^k_{Ih}(\bar{t})$, for all $h \in \mathcal{P}_{d-k} \setminus I$. The projective bundles are restrictions of the exceptional divisors in the blow–ups of $\overline{M}^k(\bar{t})$ along $\overline{M}^k_h(\bar{t})$, for all $h \in I_{d-k}$. Their quotients make up projective fibrations over $\overline{M}^k_I$:

Notation . Let $\mathbb{P}_h := \overline{M}_{0,1}(\mathbb{P}^n, |h|, 1) \times_{\mathbb{P}^n} \overline{M}^k_h$, where the fiber product is taken along evaluation functions. Note that there is a natural evaluation function at the h–th point $ev_h : \overline{M}^k_h \to \mathbb{P}^n$. Indeed, for a generic point $(C, p_1, p_h, \mathcal{L}, e)$ of $\overline{M}^k_h$, e induces a linear system on $\mathcal{L}(|h|p_h)$ whose base locus never contains p_h. There is also a natural evaluation function $ev_1 : \overline{M}_{0,1}(\mathbb{P}^n, |h|, 1) \to \mathbb{P}^n$.

$\mathbb{P}_h$ is obtained by gluing $(G_h)^{n+1}$–quotients of the projective bundles $\mathbb{P}(\mathcal{N}_{\overline{M}^k_h(\bar{t})|\overline{M}^k(\bar{t})})$. It will play an important role in the later Chow ring computations.

For any $J \subset \mathcal{P}_{d-k}$, let $\mathbb{P}_J$ denote the fiber product of all pullbacks to $\overline{M}^k_J$ of the fibrations $\mathbb{P}_h$. The morphism $f^k_J : \overline{M}^{k+1}_J \to \overline{M}^k_J$ factors into $g_J : \overline{M}^{k+1}_J \to \mathbb{P}_J$ and $\pi_J : \mathbb{P}_J \to \overline{M}^k_J$.

Reversing the order, one could also think of $\overline{M}^{k+1}_J$ as a projective fibration. There is a fibre square:

$$\begin{array}{ccc} \overline{M}^{k+1}_J & \longrightarrow & \mathbb{P}_J \\ \downarrow & & \downarrow \\ \overline{M}^{k+1}\binom{k}{J} & \longrightarrow & \overline{M}^k_J \end{array}$$

Here $\overline{M}^{k+1}\binom{k}{J}$ is the space of $(k+1)$–stable, degree $(d - |J|(d-k))$–maps from rational curves into $\mathbb{P}^n$, with a set of distinct, smooth marked points $\{p_1, \{p_h\}_{h\in J}\}$. Locally this is the $(G_J)^{n+1}$– quotient of the blow–up of $\overline{M}^k_J(\bar{t})$ along all $\overline{M}^k_{Jh}(\bar{t})$, for $h \in \mathcal{P}_{d-k}$ disjoint from the elements of J.

Remark 2.4. Consider any $h \in \mathcal{P}_{d-k} \setminus \{\emptyset, D\}$ and any nested set I such that $I \neq I \cup \{h\}$ is again a nested set. The space $\overline{M}^k_{Ih}$ maps into one component of the blow–up locus at the k–th step in $\overline{M}^k_I$. The components of the blow–up locus correspond to the different types of positionings of cardinal k subsets $h \subset D$ with respect to the elements of I.

3. The Chow ring

Previously we have found $\overline{M}_{0,1}(\mathbb{P}^n, d) =: \overline{M}^d$ to be the end product of a sequence of birational transformations. This section is concerned with the Chow ring computation of this space. The inductive method centers on writing the Chow ring of $\overline{M}_I^{k+1}$ as an algebra over that of $\overline{M}_I^k$, for any nested subset $I \subset \mathcal{P}$ and any step k. The calculation is done in two parts, each adapting methods employed for a usual blow–up (see [F], [Ke]): the classical calculation by Grothendieck of the Chow ring for a projective fibre bundle is extended to the case of a weighted projective fibration via localization. Then, the Chow ring of a weighted blow–up along a regular local embedding is related to that of its exceptional divisor via essentially a self–intersection formula. As the exceptional divisor displays multiple components at the level of étale covers, its self intersection becomes more involved and justifies the introduction of a network containing multiple copies of strata.

Localization. Here the Chow ring of a weighted projective fibration is computed as an algebra over the Chow ring of the basis, via localization. The working examples are $\overline{M}_I^1 \to \mathbb{P}^n$ and $\mathbb{P}_h \to \overline{M}_h^k$. In the first case, the underlying action of the group $(G_I)^{n+1}$ on a cover plays an important role, in the second case this group is $(S_h)^{n+1}$. The first lemma defines the properties of a weighted blow–up, going from local to global presentation.

Let G be a finite group.

Lemma 3.1. *Consider a birational morphism of reduced schemes $\pi : \tilde{X} \to X$. Suppose that X can be covered by open sets of type $U \cong U'/G$ and that $\pi^{-1}(U) \cong \tilde{U}'/G$, where $\tilde{U}'$ is the blow–up of U' along a reduced locus Y' which is left point–wise invariant by the action of G. Then*

a) $\tilde{X}$ is a weighted blow–up of X along a locus Y, namely,

$$\tilde{X} = \mathrm{Proj}(\oplus_{n\geq 0}\mathcal{I}_n)$$

for an increasing filtration $\{\mathcal{I}_n\}_{n\geq 0}$ of the ideal $\mathcal{I}_Y$, such that $\mathcal{I}_0 = \mathcal{O}_X$, $\mathcal{I}_1 = \mathcal{I}_Y$ and $\mathcal{I}_n\mathcal{I}_m \subseteq \mathcal{I}_{m+n}$ for all $m, n \geq 0$.

b) The reduced structure of the exceptional divisor in $\tilde{X}$ is

$$E = \mathrm{Proj}(\oplus_{n\geq 0}\mathcal{I}_n/\mathcal{I}_{n+1}).$$

Proof. a) Let E denote the exceptional divisor of $\pi : \tilde{X} \to X$, and Y its image in X. $\mathcal{I}_E$ and $\mathcal{I}_Y$ will denote their sheaves of ideals. The sheaves $\mathcal{I}_n := \pi_*\mathcal{I}_E^n$ form a filtration of $\mathcal{I}_Y$ such that $\mathcal{I}_0 = \mathcal{O}_X$ and $\mathcal{I}_j\mathcal{I}_k \subseteq \mathcal{I}_{j+k}$ for all indices j, k. It remains to show that $\tilde{X} = \mathrm{Proj}(\oplus_{n\geq 0}\mathcal{I}_n)$. Indeed, $\pi^{-1}(U) = \mathrm{Proj}(\oplus_{n\geq 0}(f_*\mathcal{I}_{Y'}^n)^G)$, where f is the morphism from U' to X. Also, if $\tilde{f}$ is the morphism from $\tilde{U}'$ to $\tilde{X}$, then over $\pi^{-1}(U)$, $\mathcal{I}_E^n \cong (\tilde{f}_*\mathcal{I}_{E'}^n)^G$, where E' is the exceptional divisor of $\tilde{U}'$. If $\pi' : \tilde{U}' \to U'$ is the blow–down morphism, then $(f_*\mathcal{I}_{Y'}^n)^G \cong (f_*\pi'_*\mathcal{I}_{E'}^n)^G \cong \pi_*(\mathcal{I}_E^n)_{|U} \cong \mathcal{I}_{n_{|U}}$.

In the same way, $\mathcal{I}_n/\mathcal{I}_{n+1} \cong \pi_*(\mathcal{I}_E^n/\mathcal{I}_E^{n+1})$.

b) We claim that the reduced algebra of $\mathcal{O}_Y \otimes (\oplus_{n\geq 0}\mathcal{I}_n)$ is $\oplus_{n\geq 0}\mathcal{I}_n/\mathcal{I}_{n+1}$. Let $\mathcal{J}$ be the ideal of $\mathcal{O}_Y \otimes (\oplus_{n\geq 0}\mathcal{I}_n)$ generated by the images of the morphisms $\phi_k : \mathcal{O}_Y \otimes \mathcal{I}_{k+1} \to \mathcal{O}_Y \otimes \mathcal{I}_k$. The ideal $\mathcal{J}$ is nilpotent. Indeed, let x be a section of Im ϕ_k. For any $j > k$, $x^j = x^{j-1}x$ is a section of $\mathcal{I}_{(k+1)(j-1)}\mathcal{I}_Y \subset \mathcal{I}_{kj}\mathcal{I}_Y$. On the other hand, $\mathcal{O}_Y \otimes (\oplus_{n\geq 0}\mathcal{I}_n)/\mathcal{J} \cong \oplus_{n\geq 0}\mathcal{I}_n/\mathcal{I}_{n+1}$ and $\oplus_{n\geq 0}\mathcal{I}_n/\mathcal{I}_{n+1}$ is reduced because it is locally the G–invariant part of $\oplus_{n\geq 0}\mathcal{I}^n_{Y'}/\mathcal{I}^{n+1}_{Y'}$.

□

Keeping the notations from above, further assume that each U' is a smooth scheme on which G acts as a small group, and that $Y' \subset U'$ is a regular embedding. This local situation may be expressed globally by the following properties of the filtration $\mathcal{I}_k$:

(1) $\mathcal{I}_k \cap \mathcal{I}_Y^2 = \sum_{j=1}^{k-1}\mathcal{I}_j\mathcal{I}_{k-j}$,
(2) $\mathcal{I}_k/(\mathcal{I}_k \cap \mathcal{I}_Y^2)$ is a subbundle of the conormal bundle $\mathcal{I}_Y/\mathcal{I}_Y^2$.

These requirements are the minimum that insure a natural structure of smooth Deligne–Mumford stack on $\tilde{X}$. Under these conditions, the following Lemma holds:

Lemma 3.2. *a) The normal bundle in* $A := \mathrm{Spec}(\oplus_{n\geq 0}\mathcal{I}_n/\mathcal{I}_{n+1})$ *of the fixed locus* Y *under the natural* $\mathbb{C}^*$ *action on* A *is*

$$\mathcal{N}_{Y|A} = \oplus_{n\geq 1}\mathcal{N}_n/\mathcal{N}_{n+1},$$

where $\{\mathcal{N}_n\}_n$ *is the filtration of the normal bundle* $\mathcal{N}_{Y|X}$ *dual to the filtration* $\{\mathcal{I}_n/(\mathcal{I}_n \cap \mathcal{I}_Y^2)\}_n$ *of* $\mathcal{I}_Y/\mathcal{I}_Y^2$.

b) There is a ring isomorphism

$$A^*(E;\mathbb{Q}) \cong \frac{A^*(Y;\mathbb{Q})[\tau]}{< P_{Y|X}(\tau) >},$$

where $P_{Y|X}(t)$ *is the top equivariant Chern class of the bundle* $\mathcal{N}_{Y|A}$. *In particular, the free term of* $P_{Y|X}(t)$ *is the top Chern class of* $\mathcal{N}_{Y|X}$. τ *is the first Chern class of* $\mathcal{O}_E(1)$.

Proof. a)The ideal of the zero section Y in A is $\oplus_{n\geq 1}\mathcal{I}_n/\mathcal{I}_{n+1}$. There is a natural morphism of modules over $\oplus_{n\geq 0}\mathcal{I}_n/\mathcal{I}_{n+1}$:

$$\oplus_{n\geq 1}\mathcal{I}_n/\mathcal{I}_{n+1} \to \oplus_{n\geq 1}\mathcal{I}_n/(\mathcal{I}_{n+1} + \mathcal{I}_n \cap \mathcal{I}_Y^2) \cong \oplus_{n\geq 1}\frac{\mathcal{I}_n/\mathcal{I}_n \cap \mathcal{I}_Y^2}{\mathcal{I}_{n+1}/\mathcal{I}_{n+1} \cap \mathcal{I}_Y^2}.$$

The kernel is $\oplus_{n\geq 1}(\mathcal{I}_n \cap \mathcal{I}_Y^2)/(\mathcal{I}_{n+1} \cap \mathcal{I}_Y^2)$, which by property (1) above, is isomorphic to $(\oplus_{n\geq 1}\mathcal{I}_n/\mathcal{I}_{n+1}))^2$.

b) Consider a finite G– cover $\{U'_i\}_i$ of X and $\{Y'_i\}_i$ of Y such that the blow–ups of U'_i along Y'_i form a G–cover of $\tilde{X}$. Let $\{E_i\}_i$ form the corresponding open cover of E. A canonical stratification $\{V_i\}_i$ of E is assigned: $V_i = Z_{i-1}\backslash Z_i$ where $Z_0 := E$ and $Z_i := E \setminus (\bigcup_{j=1}^{i} E_j)$ for $i \geq 1$. A standard argument based on the open–closed exact sequence of Chow groups of each

$Z_{i-1} = V_i \cup Z_i$ leads to the following generators of $A^*(E)$ as an $A^*(Y)$–module: τ^i, for all i such that $0 \leq i < e := \operatorname{codim}_X Y$. Here τ denotes $c_1(\mathcal{O}_E(1))$.

Moreover the following relations hold: $\pi_*(\tau^i) = 0$ for all $i < e$ and $\pi_*(\tau^e) = [Y]$, where π is the morphism $\pi : E \to Y$. We will now find a relation

$$\tau^e + \sum_{i=0}^{e-1} a_i \tau^i = 0 \tag{3.1}$$

on E. There is a $\mathbb{C}^*$–equivariant morphism

$$F : N_{E|\tilde{X}} := \operatorname{Spec}(\oplus_j \mathcal{I}_E^n / \mathcal{I}_E^{n+1}) \to A = \operatorname{Spec}(\oplus_{n \geq 0} \mathcal{I}_n / \mathcal{I}_{n+1})$$

for the natural $\mathbb{C}^*$ – actions having E and Y as fixed loci. An elementary form of Atiyah–Bott localization formula applied to F yields:

$$F_* \frac{1}{c_{top}^{eq}(\mathcal{N}_{E|\tilde{X}})} = \frac{1}{c_{top}^{eq}(\mathcal{N}_{Y|A})},$$

where c_{top}^{eq} denotes the equivariant top Chern class. As $c_{top}^{eq}(\mathcal{N}_{E|\tilde{X}}) = t - \tau$, it follows that $c_{top}^{eq}(\mathcal{N}_{Y|A})$ is the inverse of the Segre series $\sum_{j \geq e-1} F_*(\tau^j) t^{-j-1}$. This implies that τ is the root of the polynomial $c_{top}^{eq}(\mathcal{N}_{Y|A})$. $\square$

The above lemmas apply to any projective morphism of stacks $P \to Y$ together with a sheaf $\mathcal{L}$ on P, satisfying the following conditions: Y may be covered by open subsets Y' such that there exist a G–equivariant projective bundle $\pi' : \mathbb{P}(V') \to Y'$ and a morphism $\tilde{g} : \mathbb{P}(V') \to P$ whose image is $\mathbb{P}(V')/G$; locally $\mathcal{L}$ is the sheaf of G–invariant sections of $\mathcal{O}_{\mathbb{P}(V')}(1)$ and G acts as a small group on $\pi'_* \mathcal{O}_{\mathbb{P}(V')}(1)$. Then $P \cong \operatorname{Proj}(\oplus_n \pi_* \mathcal{L}^n)$ is the exceptional divisor of the following weighted blow–up:

$$\tilde{X} := \operatorname{Spec}(\oplus_n \mathcal{L}^n) \to X := \operatorname{Spec}(\oplus_n \pi_* \mathcal{L}^n).$$

We say that P is a weighted projective fibration.

The first application of interest to us is to the first intermediate moduli space and its substrata $\overline{M}_I^1$, for a partial partition I:

Notation . A set $I = \{h_1, ..., h_s\}$ is called a partial partition of $\{1, ..., d\}$ if $h_1, ...h_s \subset \{1, ..., d\}$ are disjoint subsets. We denote by $l_I := \sum_{i=1}^s |h_i|$.

There are two important classes on $\overline{M}_I^1$: the pullback H of the hyperplane class in $\mathbb{P}^n$ via the natural evaluation map; and the cotangent class ψ, the first Chern class of the line bundle $s_1^*(\omega_{\overline{U}_I^1 | \overline{M}_I^1})$, where $\pi : \overline{U}_I^1 \to \overline{M}_I^1$ is the universal family and s_1 is its canonical section.

Lemma 3.3. *The Chow ring with rational coefficients of $\overline{M}_I^1$ is*

$$A^*(\overline{M}_I^1; \mathbb{Q}) = \frac{\mathbb{Q}[H, \psi]}{\langle H^{n+1}, P_{\overline{M}_I^0 | \overline{M}^0}(\psi) \rangle}$$

where

$$P_{\overline{M}^0_I|\overline{M}^0}(t) = t^{s-1}\prod_{i=1}^{d-l_I}(H+it)^{n+1}.$$

Proof. Recall that $G(\mathbb{P}^n, d, 1)$ is the weighted blow–up of $\mathbb{P}^n_d$ along $\mathbb{P}^1 \times \mathbb{P}^n$ in the sense of Lemma 3.1, and $\overline{M}^1$ is the restriction of the exceptional divisor to $\{0\} \times \mathbb{P}^n$. Given any partial partition I, $\overline{M}^1_I$ is the restriction to $\{0\} \times \mathbb{P}^n$ of exceptional divisor E in the weighted blow–up of $(\mathbb{P}^1)^s \times \mathbb{P}^n_{d-l_I}$ along $\mathbb{P}^1 \times \mathbb{P}^n$. By a classical argument, pullback of the class $[E]$ from this weighted blow–up to $\overline{M}^1_I$ is $-\psi$. If the points of $\mathbb{P}^n_d, \mathbb{P}^n_{d-l_I}$ are understood modulo constants as $(n+1)$–tuples of polynomials in one variable t, then the map

$$(\mathbb{P}^1)^s \times \mathbb{P}^n_{d-l_I} \to \mathbb{P}^n_d$$

is the multiplication of each of the $(n+1)$ degree $(d-l_I)$ polynomials by the polynomial $\prod_{i=1}^s (b_i t - a_i)^{|h_i|}$, for each $\{(a_i : b_i)\}_{1\le i\le s} \in (\mathbb{P}^1)^s$. The blow–up locus $\mathbb{P}^1 \times \mathbb{P}^n$ is embedded by:

$$\mathbb{P}^1 \times \mathbb{P}^n \to (\mathbb{P}^1)^s \times \mathbb{P}^n_{d-l_I}$$

$$((a:b),(g_0 : ... : g_n)) \to (\{(a:b)\}_{1\le i\le s}, ((at-b)^{d-l_I} g_0 : ... : (at-b)^{d-l_I} g_n))).$$

The normal bundle

$$\mathcal{N}_{\mathbb{P}^1\times\mathbb{P}^n|(\mathbb{P}^1)^s\times\mathbb{P}^n_{d-l_I}}\Big|_{\mathbb{P}^n} = (\mathcal{O}^s_{\mathbb{P}^n} \oplus \oplus_{i=1}^{(n+1)(d-l_I)}\mathcal{O}_{\mathbb{P}^n}(1))/\mathcal{O}_{\mathbb{P}^n}$$

admits a natural filtration $\{\mathcal{N}_k\}_k$ described in Lemma 3.2. Here

$$\mathcal{N}_k = (\mathcal{O}^s_{\mathbb{P}^n} \oplus \oplus_{i=1}^{(n+1)(d-l_I-k)}\mathcal{O}_{\mathbb{P}^n}(1))/\mathcal{O}_{\mathbb{P}^n} = \mathcal{N}_{\mathbb{P}^1\times\mathbb{P}^n|(\mathbb{P}^1)^s\times\mathbb{P}^n_k}\Big|_{\mathbb{P}^n}.$$

$\mathbb{C}^*$ acts on the bundle $\mathcal{N} = \oplus_k \mathcal{N}_k/\mathcal{N}_{k+1}$ with weights $(1, ..., d-l_I)$, and the top equivariant Chern class of $\mathcal{N}$ is thus

$$P_{\mathbb{P}^1\times\mathbb{P}^n|(\mathbb{P}^1)^s\times\mathbb{P}^n_{d-l_I}}(t) = t^{s-1}\prod_{i=1}^{d-l_I}(H+it)^{n+1}.$$

Thus by Lemma 3.2,

$$A^*(\overline{M}^1_I;\mathbb{Q}) = \frac{\mathbb{Q}[H,\psi]}{\langle H^{n+1}, P_{\mathbb{P}^1\times\mathbb{P}^n|(\mathbb{P}^1)^s\times\mathbb{P}^n_{d-l_I}}(\psi)\rangle}.$$

Looking at the pull–back morphism $A^*(\overline{M}^1;\mathbb{Q}) \to A^*(\overline{M}^1_I;\mathbb{Q})$ we find the class of $\overline{M}^1_I$ in $\overline{M}^1$ to be a multiple of $\psi^{-s}\prod_{d-l_I+1}^d (H+i\psi)^{n+1}$. For dimension reasons, the coefficient is a constant and moreover, after push–forward to $\mathbb{P}^n$ we find the constant to be 1. Therefore

$$[\overline{M}^1_I] = \psi^{-s}\prod_{d-l_I+1}^{d}(H+i\psi)^{n+1}. \tag{3.2}$$

□

The next example where Lemma 3.2 applies is the fibration $\mathbb{P}_n \to \overline{M}^k_h$ defined in section 2. Indeed, $\mathbb{P}_h$ is the pullback to $\overline{M}^k_h$ of the weighted projective fibration $\overline{M}_{0,1}(\mathbb{P}^n, |h|, 1) \to \mathbb{P}^n$, and therefore, its Chow ring is generated over $A^*(\overline{M}^k_h)$ by the class ψ_h, the pullback of the cotangent class on $\overline{M}_{0,1}(\mathbb{P}^n, |h|, 1)$. From a different perspective, the conormal bundle of $\overline{M}^{k+1}_h$ in $\overline{M}^{k+1}$ descends to $\mathbb{P}^n$, being constant on the fibres of $g_h : \overline{M}^{k+1}_h \to \mathbb{P}_h$. The first Chern class τ_h of the descent bundle is an alternate generator of the Chow ring. Moreover,

Lemma 3.4. *There is a filtration $\{\mathcal{N}_j\}_j$ of the normal bundle $\mathcal{N}_{\overline{M}^k_h|\overline{M}^k}$ such that*

$$A^*(\mathbb{P}_h; \mathbb{Q}) = A^*(\overline{M}^k_h)[\tau_h]/ < P_h(\tau_h) >,$$

where $P_h(t)$ is the top equivariant Chern class of the fibre bundle $\mathcal{N}_h = \oplus_j \mathcal{N}_j/\mathcal{N}_{j+1}$ with respect to the natural $\mathbb{C}^*$ action.

Proof. Consider the $\bar{t}$–covers of sections 1 and 2, and the blow–up $M(\bar{t})$ of $\overline{M}^k(\bar{t})$ along $\overline{M}^k_h(\bar{t})$. The quotient by $(G_h)^{n+1}$ of the exceptional divisor is an open subset of $\mathbb{P}_h$. Following the discussion in Proposition 1.11, quotients by small subgroups in $(G_h)^{n+1}$ yield locally an étale cover of $\mathbb{P}_h$.

The quotient $M(\bar{t})/(S_h)^{n+1}$ is a weighted blow–up and thus Lemma 3.2 applies to the étale covers of $\mathbb{P}_h$, yielding the filtration $\{\mathcal{N}_j\}_j$ of the normal bundle $\mathcal{N}_{\overline{M}^k_h|\overline{M}^k}$. Indeed, for all $x \in M(\bar{t})$, the largest small subgroup H_x of the stabilizer of x decomposes into $(S_h)^{n+1} \oplus H'_x$, such that the weighted blow–up structure of $M(\bar{t})/(S_h)^{n+1}$ descends through H'_x. Functoriality of the constructions in Lemmas 3.1 and 3.2 insures the gluing property of the filtration $\{\mathcal{N}_j\}_j$ under different étale maps, and for different $\bar{t}$–s. □

The preceding examples set up the first induction steps in the calculation of $A^*(\overline{M}^k_I)$. The general step relies on the following construction:

The Chow ring associated to a network of regular local embeddings. Let $\mathcal{P}$ be a finite set and let G be a finite group acting on $\mathcal{P}$.

For every subset $I \subset \mathcal{P}$, G_I denotes the largest subgroup of G that fixes all elements of I. Among subsets of $\mathcal{P}$, we consider a special family closed under the G–action, which will be called the family of allowable sets, such that if I is allowable, then any of its subsets is allowable too.

Definition 3.5. A G– network of regular local embeddings indexed by allowable subsets I of $\mathcal{P}$ is a category of smooth Deligne–Mumford stacks $\{\overline{M}_I\}_{I \subset \mathcal{P}}$ with unique morphisms $\bar{\phi}^I_J : \overline{M}_J \to \overline{M}_I$ for every $J \supset I$, canonical isomorphisms $g : \overline{M}_I \to \overline{M}_{g(I)}$ for every $[g] \in G/G_I$, and a set of étale

covers by schemes $\overline{M}_I(t) \to \overline{M}_I$, such that there is a Cartesian diagram

$$\begin{tikzcd} \bigsqcup_{[g]\in G_I/G_J} \overline{M}_{g(J)}(t) \arrow[r] \arrow[d] & \overline{M}_I(t) \arrow[d] \\ \overline{M}_J \arrow[r, "\bar{\phi}^I_J"] & \overline{M}_I, \end{tikzcd}$$

each morphism $\overline{M}_J(t) \to \overline{M}_I(t)$ is an inclusion, $\overline{M}_J(t) \bigcap \overline{M}_K(t) = \overline{M}_{J\bigcup K}(t)$ in $\overline{M}_{J\bigcap K}(t)$, and all intersections are transverse.

We will write IJ for $I \bigcup J$ when $I \cup J$ is allowable.

The stacks $\overline{M}_I$ with $I \neq \emptyset$ minimal among the allowable subsets of $\mathcal{P}$ will be called generators of the network.

Notation . We will denote the image of $\bar{\phi}^I_J$ by $\overline{M}_I(J\backslash I)$. This depends only on the class of J modulo the action of elements in G/G_I. The morphism $\bar{\phi}^I_J$ factors through a finite map $\phi^I_J : \overline{M}_J \to \overline{M}_I(J \setminus I)$ and an embedding $j^I_J : \overline{M}_I(J\backslash I) \to \overline{M}_I$. When $I = \emptyset$ we omit the subscript I. Thus $\overline{M}$ denotes the final element of the category.

A notion of extended Chow ring is assigned to a G– network of regular local embeddings by concatenating the Chow rings of all network elements, modulo a natural equivalence relation which keeps track of the structure at the level of étale atlases:

Notation . Fix $I \subset \mathcal{P}$ allowable and $h \in \mathcal{P}$ such that $I \cup \{h\}$ is allowable. To any cycle $\alpha = [V] \in Z_l(\overline{M}_I(h))$ we will associate a cycle $\alpha_h \in Z_l(\overline{M}_{Ih})$ as follows:

$$\alpha_h = \frac{\sum_i [V^i_h]}{deg\left((\phi^I_{Ih})^{-1}(V)/V\right)}$$

where $\{V^i_h\}_i$ are the l–dimensional components of $(\phi^I_{Ih})^{-1}(V)$.

Definition 3.6. The graded $\mathbb{Q}$–vector space $B^*(\overline{M};\mathbb{Q})$ is defined as follows:

$$B^*(\overline{M};\mathbb{Q}) := \bigoplus_{l=0}^{\dim(\overline{M})} B^l(\overline{M}),$$

where the extended Chow groups are

$$B^l(\overline{M}) := \oplus_I Z^{l-\mathrm{codim}_{\overline{M}}\overline{M}_I}(\overline{M}_I)/\sim,$$

the sum taken after all allowable subsets $I \subset \mathcal{P}$ with $\mathrm{codim}_{\overline{M}}\overline{M}_I \leq l$. The equivalence relation $\sim$ is generated by rational equivalence together with relations of the type:

$$j^I_{Ih*}\alpha \sim \sum_{[g]\in G/G_h} \alpha_{g(h)},$$

for any cycle $\alpha \in Z_l(\overline{M}_I(h))$, where $\alpha_{g(h)}$ is the corresponding cycle in $\overline{M}_{Ig(h)}$.

Example 3.7. With the notations from section 2, $G := S_d$ acts on $\mathcal{P}_{>d-k} := \{h \subset \{1, ..., d\}; |h| > d - k\}$. Whenever $l \geq k - 1$, the set of strata $\overline{M}^l_I$ indexed by nested subsets $I \subset \mathcal{P}_{>d-k}$ forms an S_d– network. The associated extended Chow ring will be denoted by $B^*(\overline{M}^l, k)$. For example, $B^*(\overline{M}^k, k)$ corresponds to the network generated by normalizations of all exceptional divisors (and their strict transforms) from the steps 1 to k.

Pullbacks by the morphisms $f^l_I : \overline{M}^{l+1}_I \to \overline{M}^l_I$ are compatible with the equivalence relation $\sim$ and add up to a global pullback $f^{l*} : B^*(\overline{M}^l, k) \to B^*(\overline{M}^{l+1}, k)$.

Example 3.8. Given any G–network and a fixed member $\overline{M}_I$, the sets $\{\overline{M}_J\}_{J \supseteq I}$ and their morphisms form a G_I–network. There is a natural morphism of graded vector spaces: $B^*(\overline{M}_I) \to B^*(\overline{M})$, of degree equal to the codimension of $\overline{M}_I$ in $\overline{M}$. In addition, for $J \supset I$ there are natural pullback and pushforward morphisms $\bar{\phi}^{I*}_J : B^*(\overline{M}_I) \to B^*(\overline{M}_J)$ and $\bar{\phi}^I_{J*} : B^*(\overline{M}_J) \to B^*(\overline{M}_I)$ obtained by concatenating the usual pullbacks and pushforwards on substrata. Compatibility with $\sim$ is easily checked.

A distinctive feature coming from the étale structure of a G–network is the following “excess intersection” formula:

Lemma 3.9. *Given any allowable I, $h \in I$, and $\alpha \in A^*(\overline{M}_I(h))$, the following relation holds:*

$$\bar{\phi}^{I*}_{Ih}(j^I_{Ih*}(\alpha)) = \alpha_h \cdot c_{top}(\mathcal{N}_{\overline{M}_h|\overline{M}}) + \bar{\phi}^{Ih}_{Ihh'*}\bar{\phi}^{Ih'*}_{Ihh'}(\alpha_{h'}) \tag{3.3}$$

where $h' = g'(h)$ for some $[g'] \in G/G_h$. Here $c_{top}(\mathcal{N}_{\overline{M}_h|\overline{M}})$ denotes the top Chern class of the given normal bundle. The term $\bar{\phi}^{Ih}_{Ihh'*}\bar{\phi}^{Ih'*}_{Ihh'}(\alpha_{h'})$ does not depend on the choice of h'. This term supported on $\overline{M}_{Ih}(h') \subset \overline{M}_{Ih}$ satisfies:

$$\bar{\phi}^{Ih}_{Ihh'*}\bar{\phi}^{Ih'*}_{Ihh'}(\alpha_{h'}) \sim \sum_{[g]\neq[e]\in G/G_h} \bar{\phi}^{Ig(h)*}_{Ihg(h)}(\alpha_{g(h)}). \tag{3.4}$$

Proof. Given an l–dimensional class $j^I_{Ih*}(\alpha) = [V]$ on $\overline{M}_I$ and the preimage V_h of V in $\overline{M}_{Ih}$, the image of $[V]$ through the Gysin map $\bar{\phi}^{I*}_{Ih}$ is the intersection class of the zero section in $\mathcal{N}_{\overline{M}_{Ih}|\overline{M}_I}$ with the cone of V_h in V. By Definition 3.5, at the level of étale covers this cone has multiple components indexed by $[g] \in G/G_h$, which yield the above sum. □

Remark 3.10. For any allowable $\{h\} \cup J \subset \mathcal{P}$, there is an isomorphism of normal bundles $\mathcal{N}_{\overline{M}_{Jh}|\overline{M}_J} \cong \bar{\phi}^{h*}_{Jh}\mathcal{N}_{\overline{M}_h|\overline{M}}$. Thus for any allowable set $I \subset \mathcal{P}$,

$$\mathcal{N}_{\overline{M}_I|\overline{M}} \cong \bigoplus_{h\in I} \bar{\phi}^{h*}_I \mathcal{N}_{\overline{M}_h|\overline{M}}.$$

The next definition and lemma introduce a ring structure on $B^*(\overline{M})$.

Definition 3.11. Multiplication on $B^*(\overline{M})$ is defined as follows: given any two classes $\alpha \in A^*(\overline{M}_I)$ and $\beta \in A^*(\overline{M}_J)$, let

$$\alpha \cdot_r \beta := \bar{\phi}^{I*}_{I\cup J}(\alpha) \cdot \bar{\phi}^{J*}_{I\cup J}(\beta) \cdot c_{top}(\bar{\phi}^{I\cap J*}_{I\cup J}\mathcal{N}_{\overline{M}_{I\cap J}|\overline{M}})$$

in $A^*(\overline{M}_{I\cup J})$. Here $\bar{\phi}^{I*}_{I\cup J}, \bar{\phi}^{J*}_{I\cup J}$ are the (generalized) Gysin homomorphisms, as defined in [V].

Lemma 3.12. *$B^*(\overline{M})$ with the multiplication $\cdot_r$ admits a natural structure of graded $\mathbb{Q}-$ algebra.*

Proof. We will denote $c_{top}(\mathcal{N}_{\overline{M}_h|\overline{M}})$ by ξ_h.

Clearly the multiplication preserves the grading. The associativity is straightforward: for any $\alpha \in A^*(\overline{M}_I)$, $\beta \in A^*(\overline{M}_J)$ and $\gamma \in A^*(\overline{M}_K)$,

$$\alpha \cdot_r (\beta \cdot_r \gamma) = (\alpha \cdot_r \beta) \cdot_r \gamma =$$
$$= \alpha \cdot \beta \cdot \gamma \cdot (\textstyle\prod_{h\in I\cap J\cap K} \bar{\phi}^{h*}_{I\cup J\cup K}\xi_h) \cdot (\prod_{h\in (I\cap J)\cup(I\cap K)\cup(J\cap K)} \bar{\phi}^{h*}_{I\cup J\cup K}\xi_h).$$

The compatibility of the equivalence relation $\sim$ with the multiplication is a consequence of the previous lemma:

Consider first $j^I_{Ih*}\alpha, \beta \in A^*(\overline{M}_I)$. Via projection formula:

$$j^I_{Ih*}\alpha \cdot_r \beta = j^I_{Ih*}\alpha \cdot \beta \cdot c_{top}(\mathcal{N}_{\overline{M}_I|\overline{M}}) = j^I_{Ih*}(\alpha \cdot j^{I*}_{Ih}(\beta \cdot c_{top}(\mathcal{N}_{\overline{M}_I|\overline{M}}))) \sim$$
$$\sim \textstyle\sum_h (\alpha \cdot j^{I*}_{Ih}(\beta \cdot c_{top}(\mathcal{N}_{\overline{M}_I|\overline{M}})))_h$$

Since $\beta \cdot c_{top}(\mathcal{N}_{\overline{M}_I}|\overline{M})$ is a class on $\overline{M}_I$, the following holds:

$$(\alpha \cdot j^{I*}_{Ih}(\beta \cdot c_{top}(\mathcal{N}_{\overline{M}_I|\overline{M}})))_h = \alpha_h \cdot \bar{\phi}^{I*}_{Ih}(\beta \cdot c_{top}(\mathcal{N}_{\overline{M}_I|\overline{M}})) = \alpha_h \cdot_r \beta.$$

If $j^I_{Ih*}\alpha \in A^*(\overline{M}_I)$ and $\beta \in A^*(\overline{M}_J)$ with $I \neq J$, then by definition:

$$j^I_{Ih*}\alpha \cdot_r \beta = \bar{\phi}^{I*}_{I\cup J}(j^I_{Ih*}\alpha) \cdot \bar{\phi}^{J*}_{I\cup J}(\beta) \cdot c_{top}(\bar{\phi}^{I\cap J*}_{I\cup J}\mathcal{N}_{\overline{M}_{I\cap J}|\overline{M}}).$$

Set $\gamma = \bar{\phi}^{J*}_{I\cup J}(\beta) \cdot c_{top}(\bar{\phi}^{I\cap J*}_{I\cup J}\mathcal{N}_{\overline{M}_{I\cap J}|\overline{M}})$. The morphism $\bar{\phi}^{I*}_{I\cup J}$ may be split into a composition of pullbacks $\bar{\phi}^{I_s*}_{I_{s+1}}$, with $I_0 := I$, $I_{|J\setminus I|} = I \cup J$ and $|I_{s+1} \setminus I_s| = 1$. After successive applications of Lemma 3.9:

$$\bar{\phi}^{I*}_{I\cup J} j^I_{Ih*}\alpha = \textstyle\sum_{h\in J\setminus I} \bar{\phi}^{Ih*}_{I\cup J}\alpha_h \cdot \xi_h + \bar{\phi}^{I\cup J}_{(I\cup J)h'*}\bar{\phi}^{Ih'*}_{(I\cup J)h'}\alpha_{h'}$$

for some $h' = g(h) \notin I \cup J$. By formula (3.4),

$$\bar{\phi}^{I\cup J}_{(I\cup J)h'*}\bar{\phi}^{Ih'*}_{(I\cup J)h'}\alpha_{h'} \cdot \gamma = j^{I\cup J}_{(I\cup J)h'*}\left(\phi^{I\cup J}_{(I\cup J)h'*}\bar{\phi}^{Ih'*}_{(I\cup J)h'}\alpha_{h'} \cdot j^{I\cup J*}_{(I\cup J)h'}\gamma\right) \sim$$
$$\sim \textstyle\sum_{h'=g(h)\notin I\cup J} \bar{\phi}^{Ih'*}_{(I\cup J)h'}\alpha_{h'} \cdot \bar{\phi}^{I\cup J*}_{(I\cup J)h'}\gamma$$

Putting it all together, $j^I_{Ih*}\alpha \cdot_r \beta$ is equivalent to the following sum:

$$\textstyle\sum_{h'=g(h)\in J\setminus I} \bar{\phi}^{Ih'*}_{I\cup J}(\alpha_{h'}) \cdot \bar{\phi}^{J*}_{I\cup J}(\beta) \cdot c_{top}(\bar{\phi}^{(I\cap J)h'*}_{I\cup J}\mathcal{N}_{\overline{M}_{(I\cap J)h'}|\overline{M}})+$$
$$+\textstyle\sum_{h'=g(h)\notin J\cup I} \bar{\phi}^{Ih'*}_{(I\cup J)h'}(\alpha_{h'}) \cdot \bar{\phi}^{J*}_{(I\cup J)h'}(\beta) \cdot c_{top}(\bar{\phi}^{I\cap J*}_{(I\cup J)h'}\mathcal{N}_{\overline{M}_{I\cap J}|\overline{M}}) \sim$$
$$\sim \textstyle\sum_{h'=g(h)} \alpha_{h'} \cdot_r \beta$$

∎

Lemma 3.13. *a) Given $\alpha \in Z_l(\overline{M}_J)$ and $I \subset J$, the following relation holds:*

$$\bar{\phi}^I_{J*}(\alpha) \sim \sum_{[g] \in G_I/G_J} g_*(\alpha),$$

where $g : \overline{M}_J \to \overline{M}_{g(J)}$ is the canonical isomorphism of Definition 3.5.

b) The morphism

$$\begin{array}{ccc} Z^*(\overline{M}(h)) & \xrightarrow{e(h)} & B^*(\overline{M}) \\ \alpha & \longrightarrow & \sum_{[g] \in G/G_h} \alpha_{g(h)} \end{array}$$

factors through rational equivalence.

c) The following sequence is exact:

$$A^*(\overline{M}(h)) \xrightarrow{(j_*, -e(h))} A^*(\overline{M}) \oplus B'^*(\overline{M}) \longrightarrow B^*(\overline{M}) \to 0,$$

where $B'^*(\overline{M}) := \bigoplus_{l=0}^{\dim(\overline{M})} (\oplus_{I \neq \emptyset} Z^{l - \mathrm{codim}_{\overline{M}} \overline{M}_I}(\overline{M}_I)/\sim)$.

Proof. Consider a cycle $\alpha = [V] \in Z_l(\overline{M}(h))$. Recall the corresponding cycle $\alpha_h \in Z_l(\overline{M}_h)$ introduced for Definition 3.6. Assume that there is a sequence of distinct elements $h_1, ...h_m$ in the G–orbit Gh, such that $V \subseteq \overline{M}(h_1...h_m)$ but $V \not\subset \overline{M}(h_1...h_{m+1})$ for any $h_{m+1} \in Gh$ distinct from the previous. The morphism $\overline{M}_h(h_2...h_m) \to \overline{M}(hh_2...h_m)$ is $m:1$, which leads to the following decomposition:

$$\alpha_h = \frac{1}{m}(\alpha_h^1 + ... + \alpha_h^m)$$

(possibly $\alpha_h^i = \alpha_h^j$). The same decomposition holds for $\alpha_{g(h)}$, where by convention, $\alpha^i_{g(h)} := g_*(\alpha_h^i)$ for all $[g] \in G/G_h$.

The following equality holds in $B^*(\overline{M})$:

$$\sum_{[g] \in G/G_h} [\alpha_{g(h)}] = \sum_{[g] \in G/G_h} [\alpha^i_{g(h)}] \tag{3.5}$$

for any $i \in \{1, ..., m\}$.

Indeed, each cycle $\alpha^i_{g_1(h)} \in Z_l(\overline{M}_{g_1(h)}(g_2(h)...g_m(h))$ is equivalent via $\sim$ to a sum of cycles on $\overline{M}_{g_1(h)...g_m(h)}$, for all tuples of distinct $g_2, ..., g_m \in G/G_h$ allowable. Let $h_i := g_i(h)$. Thus:

$$\sum_{h_1} \alpha^i_{h_1} \sim \sum_{h_1, ..., h_m} \sum_{j=1}^{(m-1)!} \alpha^{i,j}_{h_1...h_m}$$

where $\alpha^{i,j}_{h_1...h_m}$ are the cycles of the l–dimensional components in $\phi^{h_1^{-1}}_{h_1...h_2}(V^i_{h_1})$, for the $(m-1)!:1$ map $\phi^{h_1}_{h_1...h_m} : \overline{M}_{h_1...h_m} \to \overline{M}_{h_1}$. The multiplicity of each component $\alpha^{i,j}_{h_1..h_m}$ in $\alpha^i_{h_1}$ is 1 since there are $(m-1)!$ permutations of $\{h_2, ..., h_m\}$. On the other hand, the étale structure of the G– network implies that for each allowable set $I = \{h_1, ..., h_m\} \subset Gh$, the preimages

$\bigcup_{h_1}(\phi_I^{h_1})^{-1}(V^i_{h_1})$ and $\phi_I^{-1}(V)$ have the same l–dimensional components, with same multiplicities. Thus we have shown that $\sum_{h_1}\alpha^i_{h_1}$ does not depend on i, which proves the claim.

Claims a) and b) follow from the previous relation. Indeed, given $\alpha \in Z_l(\overline{M}_J)$, it is enough to notice that $(\phi^I_{J*}\alpha)_h$ contains α as a summand of its components for any $h \in J \setminus I$, and then apply $\sim$ and relation (3.5) to obtain a).

For b), consider a cycle $\alpha \in Z_l(\overline{M}(h))$ rationally equivalent to 0. We may assume for our purposes that $\alpha = [div(r)]$ for a rational function r on a $(l+1)$–dimensional scheme $W \subset \overline{M}(h)$. If we denote by r_h the restriction of $r \circ \phi_h$ to the $(l+1)$–dimensional components of $\phi_h^{-1}(W)$, then $r_h^{-1}(0)$ may capture only some of the l–dimensional components of $\phi_h^{-1}(r^{-1}(0))$. However, via relation (3.5),

$$\sum_h [r^{-1}(0)]_h \sim \sum_h [r_h^{-1}(0)]$$

and the same is true for ∞. Since $[div(r_h)] = 0$ in $A_l(\overline{M}_h)$, it follows that $\sum_h [div(r)]_h = 0$ in $B(\overline{M})$. This finishes the proof of b). c) is a direct consequence of b) and the definition of B^*. □

The usual Chow ring of $\overline{M}$ may be recovered as invariant subring of $B^*(\overline{M})$:

Lemma 3.14. *The action of G on $\mathcal{P}$ induces a canonical action on $B^*(\overline{M})$. Then*

$$A^*(\overline{M}) = B^*(\overline{M})^G.$$

Proof. By definition, the image of the natural morphism of $A^*(\overline{M}) \to B^*(\overline{M})$ is clearly included in $B^*(\overline{M})^G$. A right inverse of this morphism is given by the global push–forward morphism $\Phi : B^*(\overline{M}) \to A^*(\overline{M})$. For $\alpha \in A_l(\overline{M}_I)$, let

$$\Phi(\alpha) := k_{|I|}\bar{\phi}_{I*}\alpha,$$

where $k_I := \frac{|G_I|}{|G|}$ is chosen such that Φ is compatible with the equivalence relation in $B^*(\overline{M})$. Φ plays the role of Reynolds operator: it is clearly invariant with respect to the G–action, and Lemma 3.13, a) implies that $\Phi(\alpha) = \alpha$ for any $\alpha \in B^*(\overline{M})$. □

The extended Chow rings of the moduli spaces $\overline{M}^k$ and their substrata. are computed here inductively after the order of blow–ups. At step $k+1$, and for fixed $I \subset \mathcal{P}$, the ring $B^*(\overline{M}^{k+1}_I, k+1)$ is expressed as an algebra over $B^*(\overline{M}^k_I, k)$. We recall that $B^*(\overline{M}^k_I, k)$ is the extended Chow ring of the network generated by all normalizations of the exceptional divisors of $\overline{M}^k_I$ from previous blow–up steps. An intermediate algebra is $B^*(\overline{M}^k_I, k+1)$, whose network has the $(k+1)$–th blow–up loci as additional generators. Lemma 3.15 expresses $B^*(\overline{M}^k_I, k+1)$ as an algebra over

$B^*(\overline{M}_I^k, k)$ and Lemma 3.19 describes the algebra $B^*(\overline{M}_I^{k+1}, k+1)$ over $B^*(\overline{M}_I^k, k+1)$. For this we remember that modulo an equivalence relation $\sim$, $B^*(\overline{M}_I^{k+1}, k+1)$ is made by summands $B^{*+|J\setminus I|}(\overline{M}_J^{k+1}, k)$ for all allowable sets J such that $I \subseteq J \subset \mathcal{P}$ and $J \setminus I \subset \mathcal{P}_{d-k}$; similarly $B^*(\overline{M}_I^k, k+1)$ is made of $B^{*+|J\setminus I|e}(\overline{M}_J^k, k)$ for the same set of J–s. Lemma 3.16 identifies generators for the modules $B^*(\overline{M}_J^{k+1}, k)$ over $B^*(\overline{M}_J^k, k)$, while a suitable pullback compatible with $\sim$ is written in Lemma 3.17. The calculations are hardly surprising – they essentially transpose the usual Chow ring computations of [F], [Ke] into our context.

A thread of induction goes through Lemmas 3.15 –3.19: we assume them true at all previous blow–up steps and for all $J \supset I$. The Lemmas are trivial at step 1, and for J large enough that there is no element $h \in \mathcal{P}_{\geq d-k} \setminus J$ compatible with J, Lemma 3.4 applies since then $\overline{M}_J^{k+1} = \mathbb{P}_J$.

Notation . We will employ the same notation $T_h := -[\overline{M}_{Ih}^l] \in A^0(\overline{M}_{Ih}^l)$, for all $l > d - |h|$. The first Chern class of the normal line bundle $\mathcal{N}_{\overline{M}_{Ih}^l|\overline{M}_I^l}$ will be denoted by $-\tau_h$.

We will omit the superscript l for morphisms between l–th intermediate spaces, as it will be visible in the superscripts of the domains and codomains. In particular, the blow–down morphism $\overline{M}_I^{k+1} \to \overline{M}_I^k$ will be denoted by f_I.

Lemma 3.15. *Assume that the pullback morphisms $\bar{\phi}_J^{I*} : B^*(\overline{M}_I^k, k) \to B^*(\overline{M}_J^k, k)$ are surjective for any $I \subset J$ such that $J \setminus I \subset \mathcal{P}_{\geq d-k}$. Then the following $\mathbb{Q}$–algebra isomorphism holds:*

$$B^*(\overline{M}_I^k, k+1) \cong B^*(\overline{M}_I^k, k)[\{y_h\}_{h \in \mathcal{P}_{d-k} \setminus I}]/\mathcal{I},$$

where the ideal $\mathcal{I}$ is generated by:

1. *$y_h^2 = y_h a$, where $a \in B^*(\overline{M}_I^k, k)$ is a class whose pullback to each $\overline{M}_{Ih}^k$ is the first Chern class of the normal bundle $\mathcal{N}_{\overline{M}_h^k|\overline{M}^k}$;*
2. *$y_h y_{h'}$ unless $h \cap h' = h, h'$ or $\emptyset$;*
3. *$\prod_{h \in J \setminus I} y_h \ker \bar{\phi}_J^{I*}$, for any J as above;*
4. *$(\sum_{h \in \mathcal{P}_{d-k} \setminus I, h \cap h' = \emptyset} y_h) T_{h'} = -[\overline{M}_{Ih'}^k(h)]$ for any $h' \in \mathcal{P}_{>d-k} \cup I$, where $\overline{M}_{Ih'}^k(h) \hookrightarrow \overline{M}_{Ih'}^k$ is the image of $\overline{M}_{Ihh'}^k$ in $\overline{M}_{Ih'}^k$, for any choice of $h \in \mathcal{P}_{d-k} \setminus I$ such that $h \cap h' = \emptyset$. Here by convention the formula also holds for $h' = \emptyset$, where $T_\emptyset$ is the unit element in $B^*(\overline{M}_I^k, k+1)$.*

Proof. A morphism $F : B^*(\overline{M}_I^k, k+1) \to B^*(\overline{M}_I^k, k)[\{y_h\}_{h \in \mathcal{P}_{d-k} \setminus I}]$ is defined on the generators via the isomorphism

$$B^j(\overline{M}_I^k, k+1) \cong \bigoplus_{J \supseteq I, J \setminus I \subset \mathcal{P}_{d-k}} B^{j-|J\setminus I|e}(\overline{M}_J^k, k)/\sim,$$

where $e = \operatorname{codim}_{\overline{M}^k} \overline{M}^k_h = (n+1)(d-k)+1$, and the equivalence relation $\sim$ previously defined holds for $h \in \mathcal{P}_{d-k} \setminus I$.

Given J as above and $\alpha_J \in B^l(\overline{M}^k_J, k)$, there exists $\tilde{\alpha}_J$ in $B^{l+|J\setminus I|e}(\overline{M}^k_I, k)$ such that $\alpha_J = \bar{\phi}^{I*}_J \tilde{\alpha}_J$. Define $F(\alpha_J) := \tilde{\alpha}_J \prod_{h\in J\setminus I} y_h$. Relation (3) insures that the definition does not depend on the choice of the class $\tilde{\alpha}_J$. Moreover, compatibility with $\sim$ reduces to relation (4) via the surjectivity of $\bar{\phi}^{I*}_{Jh}$ for any $h \in \mathcal{P}_{d-k}$. Finally, relation (1) makes F into a morphism of rings. The inverse morphism is identity on $B^*(\overline{M}^k_I, k)$ and takes y_h to the class $[\overline{M}^k_{Ih}] \in B^0(\overline{M}^k_{Ih}, k)$.

□

Lemma 3.16. *The generators of the $B^*(\overline{M}^k_I, k)$–module $B^*(\overline{M}^{k+1}_I, k)$ are linear combinations of products $\prod_{h\in J\setminus I} T_h^{l_h}$, for all allowable $J \supseteq I$ such that $J \setminus I \subset \mathcal{P}_{d-k}$, and $0 \le l_h \le e$.*

Proof. The usual open–closed exact sequence implies the surjectivity of the map between graded modules:

$$f^*B^*(\overline{M}^k_I, k) \oplus B^*(\overline{M}^{k+1}_I(h), k) \to B^*(\overline{M}^{k+1}_I, k),$$

for $h \in \mathcal{P}_{d-k} \setminus I$. Any generator β of $j^I_{Ih*}B^*(\overline{M}^{k+1}_I(h), k)$ is an element of $j^J_{Jh*}A^*(\overline{M}^{k+1}_J(h))$ for some nested set $J \supseteq I$ such that $J \setminus I \subset \mathcal{P}_{>d-k}$, therefore being invariant under the action of the group $G_{J\setminus I} \subset S_d$ which leaves all elements of $J \setminus I$ invariant. On the other hand, $j^J_{Jh*}B^*(\overline{M}^{k+1}_J(h), k)$ is embedded in $B'^*(\overline{M}^{k+1}_J, k+1)$, which is generated by $B^{*+e|K\setminus J|}(\overline{M}^{k+1}_K, k)$, for all $K \supset J$ such that $K \setminus J \subset \mathcal{P}_{d-k}$. By induction, generators of $B^*(\overline{M}^{k+1}_K, k)$ are written over $B^*(\overline{M}^k_K, k)$ as polynomials in $\{T_h\}_{h\in\mathcal{P}_{d-k}}$. It follows that β has a polynomial expression of variables $\{y_h\}$ and $\{T_h\}$ over $B^*(\overline{M}^k_J, k)$, for all $h \in \mathcal{P}_{d-k}$, the exponents of T_h not exceeding e by Lemma 3.4. Let $a \in B^e(\overline{M}^k, k)$ be a class whose pullback to $\overline{M}^k_h$ is $c_{top}(\mathcal{N}_{\overline{M}^k_h|\overline{M}^k})$. Invariance to $G_{J\setminus I}$, combined with relations

$$y_h^2 = y_h a \text{ and } T_h y_h = T_h a$$

imply that β has in fact a polynomial expression of variables $\{T_h\}$ and $\{y_J(h) = \sum_{h'} y_{h'}\}$, over $B^*(\overline{M}^k_J, k)$, where $h \in \mathcal{P}_{d-k}$ and the sum is taken after all $h' \in \mathcal{P}_{d-k}$ having the same set of incidence relations with the elements of J as h. This concludes the proof, as $y_J(h) \in B^*(\overline{M}^k_J, k)$. We note that β also exhibits invariance to $G_{J\setminus I}$ as a polynomial in $\{T_h\}$, but this condition does not concern us here. It will be useful for Corollary 3.18.

□

Notation . Set $e := (n+1)(d-k)+1$. We will denote by $P_{\overline{M}^k_h|\overline{M}^k}(t) \in B^*(\overline{M}^k_I, k)[t]$ a fixed polynomial

$$P_{\overline{M}^k_h|\overline{M}^k}(t) = t^e + \sum_{i=0}^{e-1} a_{h,i} t^i$$

whose pullback to $B^*(\overline{M}^k_{Ih}, k)[t]$ is P_h, the polynomial of Lemma 3.4, and whose free term is y_h.

Define $Q_h(t) := \frac{P_{\overline{M}^k_h|\overline{M}^k}(0) - P_{\overline{M}^k_h|\overline{M}^k}(t)}{t}$.

τ_h will denote the first Chern class of the conormal line bundle $\mathcal{N}^\vee_{\overline{M}^{k+1}_h|\overline{M}^{k+1}}$, as well as its pull–backs to any $\overline{M}^{k+1}_{Ih}$. Recall that $\tau_h^i = -T_h^{i+1}$ by the multiplication rule in $B^*(\overline{M}^{k+1}, k+1)$. Thus when $h \in I$, τ_h and T_h will designate the same class on $\overline{M}^{k+1}_I$.

Lemma 3.17. *Assume that the pullback morphisms $\bar{\phi}^{I*}_J : B^*(\overline{M}^k_I, k) \to B^*(\overline{M}^k_J, k)$ are surjective for any $I \subset J$ such that $J \setminus I \subset \mathcal{P}_{d-k}$. Then the morphisms of $\mathbb{Q}$–vector spaces $\bar{f}^*_J : B^*(\overline{M}^k_J, k) \to B^*(\overline{M}^{k+1}_J, k)$,*

$$\bar{f}^*_J(\alpha) := f^*_J(\alpha) \prod_{h \in J} Q_h(\tau_h)$$

for all allowable sets J as above add to a global pullback

$$\bar{f}^*_I : B^*(\overline{M}^k_I, k+1) \to B^*(\overline{M}^{k+1}_I, k+1).$$

Proof. Due to the surjectivity of the maps $\bar{\phi}^{I*}_J$, compatibility with the equivalence relation $\sim$ reduces to the following fact: given $J \supset I$ as above, and given $y := [\overline{M}^k_J(h)] \in A^0(\overline{M}^k_J(h))$ and $y_h := [\overline{M}^k_{Jh}] \in A^0(\overline{M}^k_{Jh})$ such that $j^J_{Jh*} y = \sum_h y_h$ in $B^*(\overline{M}^k_I, k+1)$, then

$$f^*_J j^J_{Jh*} y = \sum_h Q_h(\tau_h) \tag{3.6}$$

in $B^*(\overline{M}^{k+1}_I, k+1)$. This is proved by decreasing induction on $|J|$. Indeed, $f^*_J j^J_{Jh*} y = j^J_{Jh*}\alpha = \sum_h \alpha_h$ for some class $\alpha \in A^0(\overline{M}^{k+1}_J(h))$. Moreover by the previous lemma, $j^J_{Jh*}\alpha$ may be written as a polynomial of variables $T_h := -[\overline{M}^{k+1}_{Jh}] \in A^*(\overline{M}^{k+1}_{Jh})$ over $B^*(\overline{M}^k_J, k)$, of degree no larger than e in each variable. Moreover, for dimension reasons, T_h^e can only appear in $j^J_{Jh*}\alpha$ with a numerical coefficient b_h. In fact, $b_h = -1$, as can be seen via the push–forward $f_{h*} : B^*(\overline{M}^{k+1}_{Jh}, k+1) \to B^*(\overline{M}^{k+1}_J(^k_h), k+1)$ (as in the end of section 2), because f_{h*} maps each monomial $T_h^{l_h}$ into zero whenever $l_h < e$, and maps T_h^e into y_h. It remains to find the other coefficients.

Pullback to $\overline{M}_{Jh}^{k+1}$ and the intersection formula of Lemma 3.9 yield, on the one hand:

$$\bar{\phi}_{Jh}^{J*} j_{Jh*}^{J} \alpha = \bar{\phi}_{Jh}^{J*} f_J^* j_{Jh*}^{J} y = f_{Jh}^* \bar{\phi}_{Jh}^{J*} j_{Jh*}^{J} y = f_{Jh}^* (c_e(\mathcal{N}_{\overline{M}_h^k|\overline{M}^k}) + y_h(h'))$$

where $y_h(h') = \bar{\phi}_{Jhh'*}^{Jh} \bar{\phi}_{Jhh'}^{Jh'*} y_{h'}$ is the class of the blow–up locus in $\overline{M}_{Jh}^k$. As a polynomial over $B^*(\overline{M}_{Jh}^{k+1}, k+1)$, $\bar{\phi}_{Jh}^{J*} j_{Jh*}^{J} \alpha$ preserves the form of $j_{Jh*}^{J} \alpha$. Pullback from $\mathbb{P}_h$ of relation in Lemma 3.4 yields

$$c_e(\mathcal{N}_{\overline{M}_h^k|\overline{M}^k}) = \tau_h Q(\tau_h).$$

By induction, $f_{Jh}^* y_h(h') = \sum_{h'} T_{h'} Q(T_{h'})$, the sum being after all $h' \in \mathcal{P}_{d-k} \setminus Jh$ compatible with h. Therefore in $B^*(\overline{M}_J^{k+1}, k+1)$,

$$j_{Jh*}^{J} \alpha = T_h Q(T_h) + \sum_{h'} T_{h'} Q(T_{h'}),$$

modulo $\ker \bar{\phi}_h^* : B^*(\overline{M}^k, k) \to B^*(\overline{M}_h^k, k)$. Applied for all $h \in \mathcal{P}_{d-k}$, this completely determines the coefficients of α as a polynomial in $\{T_h\}_h$. This ends the compatibility check.

It remains to prove that $\bar{f}_I^*$ preserves the multiplication structure. Given the definition of $\bar{f}_I^*$, the only relevant case is that of $\bar{f}_I^*(y_h^2)$, when $y_h := [\overline{M}_{Ih}^k] \in A^0(\overline{M}_{Ih}^k)$. Then $y_h^2 = a y_h$, where $a \in B^*(\overline{M}^k, k)$ is a class whose pullback to $B^*(\overline{M}_h^k, k)$ is $c_e(\mathcal{N}_{\overline{M}_h^k|\overline{M}^k})$. Thus $\bar{f}_I^*(y_h^2) = -T_h Q(T_h) a$. On the other hand, $\bar{f}_I^*(y_h) \bar{f}_I^*(y_h) = -T_h^2 Q(T_h)^2$, which equals the former expression since $\tau_h Q(\tau_h) = c_e(\mathcal{N}_{\overline{M}_h^k|\overline{M}^k})$ on $\overline{M}_{Ih}^k$.

□

Corollary 3.18. *An element $\beta \in B^*(\overline{M}_I^{k+1}, k)$ is zero if and only if $T_h \beta = 0$ for all $h \in \mathcal{P}_{d-k}$ and $f_{I*}(\beta) = 0$ in $B^*(\overline{M}_I^k, k)$.*

Proof. Recall that the map f_I factors through $g_I : \overline{M}_I^{k+1} \to \mathbb{P}_I$ and $p_I : \mathbb{P}_I \to \overline{M}_I^k$. On the weighted projective fibration $p_I : \mathbb{P}_I \to \overline{M}_I^k$ the result is trivial. Thus we may assume $g_{I*}(\beta) = 0$.

By Lemma 3.16, an element $\beta \in B^*(\overline{M}_I^{k+1}, k)$ can be written as a polynomial of variables $\{T_h\}$, $h \in \mathcal{P}_{d-k}$ over $B^*(\overline{M}_I^k, k)$, the exponents of T_h not exceeding e. This can be refined to exponents strictly less than e, due to the invariance of β under some group $G_{J \setminus I}$, discussed in the proof of Lemma 3.16, and to relation (3.6) in Lemma 3.17. Consequently, $g_{I*}(\beta)$ is the term free of variables $\{T_h\}_{h \notin I}$.

Fix $h \notin I$. The pullback $\bar{\phi}_{Ih}^{I*} \beta$ has the same polynomial expression as β, but over $B^*(\overline{M}_{Ih}^k, k)$. As $\overline{M}_{Ih}^{k+1}$ is a weighted projective fibration in the h–direction, the annihilation of $T_h \beta$ implies that β, as a polynomial in T_h, has all coefficients in $\ker \bar{\phi}_{Ih}^{I*} \subset B^*(\overline{M}_I^k, k)$ (see Lemma 3.4). Inductively we

reason that the coefficient of $\prod_{h\in J\setminus I} T_h^{l_h}$ in β is an element of $\ker \bar\phi_J^{I*}$, for each allowable J with $J\setminus I \subset \mathcal{P}_{d-k}$. This concludes the proof, as the relation

$$\prod_{h\in J\setminus I} T_h^{l_h} \ker \bar\phi_J^{I*} = 0$$

is inherent in the multiplicative structure of $B^*(\overline{M}_I^{k+1}, k+1)$.

□

Lemma 3.19. *Under the same conditions as in the preceding lemmas, the following $\mathbb{Q}$–algebra isomorphism holds:*

$$B^*(\overline{M}_I^{k+1}, k+1) \cong B^*(\overline{M}_I^{k}, k+1)[\{T_h\}_{h\in\mathcal{P}_{d-k}}]/\mathcal{J},$$

where the ideal $\mathcal{J}$ is generated by:

(1) $T_hT_{h'}$ *unless* $h\cap h' = h, h'$ *or* $\emptyset$*;*
(2) $\prod_{h\in J\setminus I} T_h \ker\bar\phi_J^{I*}$*, for any* $J\supseteq I$ *such that* $J\setminus I\subset \mathcal{P}_{d-k}$*;*
(3) $P_{\overline{M}_h^k|\overline{M}^k}(T_h)$*, for all* $h\in\mathcal{P}_{d-k}$*.*

Proof. We denote by $QB^*(\overline{M}_I^{k+1}, k{+}1)$ the $B^*(\overline{M}_I^{k}, k{+}1)$– algebra $B^*(\overline{M}_I^{k}, k+1)[\{T_h\}_{h\in\mathcal{P}_{d-k}}]/\mathcal{J}$. There is an obvious morphism of algebras:

$$\Phi : QB^*(\overline{M}_I^{k+1}, k+1) \to B^*(\overline{M}_I^{k+1}, k+1),$$

sending each generator T_h into the class $-[\overline{M}_{Ih}^{k+1}] \in B^0(\overline{M}_{Ih}^{k+1}, k)$. Indeed, by Lemma 3.17 and the multiplicative structure of $B^*(\overline{M}_I^{k+1}, k+1)$, all generators of $\mathcal{J}$ map to zero in $B^*(\overline{M}_I^{k+1}, k+1)$. It remains to construct the inverse of Φ.

Via the surjective map $f_I^*B^*(\overline{M}_I^{k}, k)\oplus B^*(\overline{M}_I^{k+1}(h), k) \to B^*(\overline{M}_I^{k+1}, k)$, the exact sequence of Lemma 3.13, c) can be further refined to

$$BK^*(\overline{M}_I^{k+1}, k) \to f_I^*B^*(\overline{M}_I^{k}, k)\oplus B'^*(\overline{M}_I^{k+1}, k+1) \to B^*(\overline{M}_I^{k+1}, k+1)$$

where $BK^l(\overline{M}_I^{k+1}, k) = \{\alpha \in B^l(\overline{M}_I^{k+1}(h), k) | j_{I*}\alpha \in f_I^* j_{I*} B^*(\overline{M}_I^{k}(h), k)\}$. Let the rightmost morphism above be Ψ. There is a natural morphism of $B^*(\overline{M}_I^{k}, k)$–modules

$$F : B^*(\overline{M}_I^{k}, k)\oplus B'^*(\overline{M}_I^{k+1}, k+1) \to QB^*(\overline{M}_I^{k+1}, k+1).$$

Indeed, $B'^*(\overline{M}_I^{k+1}, k{+}1)$ is generated by the $B^*(\overline{M}_J^{k}, k)$–modules $B^*(\overline{M}_J^{k+1}, k)$, for $J\setminus I\subset\mathcal{P}_{d-k}$, which are generated by monomials $\prod_{h\in J} T_h^{l_h}$, and we may simply define

$$F(\prod_{h\in J} T_h^{l_h}) := \prod_{h\in J} T_h^{l_h}.$$

The image of $B^*(\overline{M}_I^{k}, k)$ in $QB^*(\overline{M}_I^{k+1}, k+1)$ is naturally defined via the inclusion $B^*(\overline{M}_I^{k}, k) \hookrightarrow B^*(\overline{M}_I^{k}, k{+}1)$. The construction of F implies $\Phi\circ F = \Psi$. It remains to show that the image of $BK^*(\overline{M}_I^{k+1}, k)$ through F is zero.

We recall that $j_{I*}B^*(\overline{M}^k_I(h),k) \hookrightarrow B'^*(\overline{M}^k_I, k+1)$ is generated over $B^*(\overline{M}^k_I, k)$ by classes $[y_h]$ with $h \in \mathcal{P}_{d-k}$. Thus the elements in the image of the morphism

$$BK^*(\overline{M}^{k+1}_I, k) \to f_I^* B^*(\overline{M}^k_I, k) \oplus B'^*(\overline{M}^{k+1}_I, k+1)$$

are linear combinations of $(y_h, -\{Q(T_h)T_h\}_{h\in\mathcal{P}_{d-k}})$ for $h \in \mathcal{P}_{d-k}$, and of elements in $(0, \ker j_{I*})$ over $B^*(\overline{M}^k_I, k+1)$. The first are mapped to zero in $QB^*(\overline{M}^{k+1}, k+1)$ by relation (3) in the ring above, and the second type of elements are also mapped into zero by the same proof as of Corollary 3.18, because of relation (2).

□

The following lemma is an adaptation to our situation of Lemma 5.2 from [FM]. It presents an inductive way of constructing the classes of the blow–up loci, by showing how such classes are transformed under the preceding steps of weighted blow–up.

Fix $I, J \subset \mathcal{P}$ such that $I \cap J = \emptyset$. Recall that $\overline{M}^l_I(J)$ is the image of $\bar{\phi}^I_{I\cup J} : \overline{M}^l_{I\cup J} \to \overline{M}^l_I$. Here we consider the case $J = \{h'\}$, and $|h'| = d-k' < d-k$.

Lemma 3.20. *The class of the strict transform $\overline{M}^{k+1}_I(h')$ of $\overline{M}^k_I(h')$ in $B^*(\overline{M}^{k+1}_I, k+1)$ is given by*

$$[\overline{M}^{k+1}_I(h')] = f_I^*([\overline{M}^k_I(h')]) + \sum_{h\in\mathcal{P}_{d-k}} (P^h_{\overline{M}^k_{h'}|\overline{M}^k}(T_h) - P^h_{\overline{M}^k_{h'}|\overline{M}^k}(0))$$

where h' is a subset of h and $P^h_{\overline{M}^k_{h'}|\overline{M}^k}(t)$ is a polynomial over $B^*(\overline{M}^k_I, k)$ whose pull–back to $\overline{M}^k_{Ih}$ is $P_{\overline{M}^k_{Ih}|\overline{M}^k_I}(t)/P_{\overline{M}^k_{Ih}|\overline{M}^k_{Ih'}}(t)$.

Note that this polynomial does not depend on the choice of $h' \in \mathcal{P}_{d-k'}$, as long as $h' \subset h$, and neither do the classes $[\overline{M}^k_I(h')]$ and $[\overline{M}^{k+1}_I(h')]$, as the group G_h permutes the elements of $h' \in \mathcal{P}_{d-k'}$.

Proof. By Corollary 3.18, it is enough to show that the above formula holds when we apply f_{I*} and $\bar{\phi}^{I*}_{Ih}$ to both sides. The first part is a consequence of the equality

$$f_{I*}(P^h_{\overline{M}^k_{h'}|\overline{M}^k}(T_h) - P^h_{\overline{M}^k_{h'}|\overline{M}^k}(0)) = 0,$$

since the polynomial above has no free terms in T_h. The second part comes out of descending induction on $|I|$.

The class of the preimage $(\bar{\phi}^I_{Ih})^{-1}(\overline{M}^k_I(h'))$ is the sum of two distinct components: a constant multiple of $[\overline{M}^k_{Ih}]$, corresponding to the set $\{h' \in \mathcal{P}_{d-k'} | h' \subset h\}$, and $[\overline{M}^k_{Ih}(h')]$ corresponding to the set $\{h' \in \mathcal{P}_{d-k'} | h' \cap h = \emptyset\}$. Since in the last case, the preimages of $\overline{M}^k_{Ih'}$ and $\overline{M}^k_{Ih}$ intersect transversely in an étale cover of $\overline{M}^k_I$, a standard dimension argument shows that

$f_{Ih}^*[\overline{M}^k_{Ih}(h')] = [\overline{M}^{k+1}_{Ih}(h')]$. For the former case, the induction hypothesis on $\overline{M}^{k+1}_{Ih}$ states that

$$[\overline{M}^{k+1}_{Ih}(h')] = g_{Ih}^*[\mathbb{P}_h(h')] + \sum_{j\in\mathcal{P}_{d-k}} (P^j_{\overline{M}^k_{j'}|\overline{M}^k}(T_j) - P^j_{\overline{M}^k_{j'}|\overline{M}^k}(0))$$

where the sum is taken after all j with $\{j,h\}$ allowable, and $j' \subset j$ is fixed for each j. Here $\mathbb{P}_h(h')$ is the weighted projective fibration of $\mathcal{N}_{\overline{M}^k_h|\overline{M}^k_{h'}}$, pulled back to $\overline{M}^k_{Ih}$. It is locally the quotient $\bigcup_{h'\subset h}\mathbb{P}(\mathcal{N}_{\overline{M}^k_{Ih}(\bar{t})|\overline{M}^k_{Ih'}(\bar{t})})/G_{Ih}$. The morphism $f_{Ih}: \overline{M}^{k+1}_{Ih} \to \overline{M}^k_{Ih}$ factors through $g_{Ih}: \overline{M}^{k+1}_{Ih} \to \mathbb{P}_h \times_{\overline{M}^k_h} \overline{M}^k_{Ih}$ and $\pi_h : \mathbb{P}_h \times_{\overline{M}^k_h} \overline{M}^k_{Ih} \to \overline{M}^k_{Ih}$, while $g_{Ih}(\overline{M}^{k+1}_{Ih}(h')) = \mathbb{P}_h(h')$.

It remains to show that the class of $\mathbb{P}_h(h')$ in $\mathbb{P}_h \times_{\overline{M}^k_h} \overline{M}^k_{Ih}$ is

$$[\mathbb{P}_h(h')] = P^h_{\overline{M}^k_{h'}|\overline{M}^k}(T_h) - P^h_{\overline{M}^k_{h'}|\overline{M}^k}(0).$$

This is proven by the same methods as in Lemma 3.2: Let τ_h be the class of the line bundle $\mathcal{O}_{\mathbb{P}_h}(1)$. After writing the class $[\mathbb{P}_h(h')]$ as

$$[\mathbb{P}_h(h')] = \sum_{i=0}^{e-1} b_i\tau_h^i,$$

where $e = \operatorname{codim}_{\overline{M}^k} \overline{M}^k_h$, and considering $\pi_{h*}([\mathbb{P}_h(h')]\tau_h^l)$ for all l, we obtain that the sum $\sum_i b_i t^i$ satisfies:

$$(\sum_i b_i t^i)(\sum_i \pi_{h*}(\tau_h^i)t^{-i-1}) = \sum_i \pi_{h*}([\mathbb{P}_h(h')]\tau_h^i)t^{-i-1}.$$

The right side is the equivariant Segre polynomial of $\mathcal{N}_{\overline{M}^k_h|\overline{M}^k_{h'}}$ in the sense of Lemma 3.2 , i.e. the inverse of $P_{\overline{M}^k_h|\overline{M}^k_{h'}}$, and $\pi_{h*}(\tau_h^i)t^{-i-1}$ is the inverse of $P_{\overline{M}^k_h|\overline{M}^k}$. This finishes the proof of the lemma. □

It remains to compute the polynomials $P_{\overline{M}^k_{Kh}|\overline{M}^k_K}$, for a nested set K and $h \in \mathcal{P}_{d-k}\setminus K$. The single important case is when K is a partial partition. We may split K into two partial partitions I and J, one containing subsets of h, the other containing sets disjoint from h.

Notation . Consider two partial partitions I and J of M_0, such that $IJ := I\bigcup J$ is still a partial partition. Let $h_0 \in \mathcal{P}_{d-k_0}$ such that $h_0 \supseteq \cup_{h\in I} h$. To these and to any number k such that $k_0 \le k \le d - |\cup_{h\in I} h|$, we associate the polynomial $P^k_{I,J}(\{t_h\}_{h\supseteq h_0})$ of variables $\{t_h\}_{h\supseteq h_0}$ over $\mathbb{Q}[H,\psi]$, defined as follows:

$$P^k_{I,J}(\{t_h\}_{h\supseteq h_0}) = \tag{3.7}$$

$$= \psi_{h_0}^{|I|-1}\Big(\prod_{j=1+|h_0|-(d-k)}^{|h_0\setminus\cup_{h\in I}h|}(H_{h_0}+j\psi_{h_0})^{n+1} - H_{h_0}^{(n+1)(d-k-|\cup_{h\in I}h|)}\Big),$$

where

$$\psi_{h_0} = \psi + \sum_{h\supseteq h_0} t_h,$$

$$H_{h_0} = H + (d - |h_0\cup(\cup_{h\in J}h)|)\psi + \sum_{h\supset h_0}|h\setminus(h_0\cup(\cup_{h'\in J}h'))|t_h.$$

We denote by

$$(3.8)\quad P_{I,J}^k(0) := \psi^{|I|-1}\prod_{j=1}^{d-k-|\cup_{h\in I}h|}(H+(k-|\cup_{h\in J}h|+j)\psi)^{n+1}.$$

$P_{I,J}^k$ admits the following geometrical interpretation:

Lemma 3.21. *Let $h\in\mathcal{P}_{\geq d-k}$ be such that $h\supset\bigcup_{h'\in I}h'$ and $h\bigcap h'=\emptyset$ for all $h'\in J$. Then the following relation holds:*

$$P_{\overline{M}^k_{Jh}|\overline{M}^k_{IJ}}(t) = P^k_{I,J}(\{T_{h'}\}_{h'\supset h}, t).$$

The class of $[\overline{M}^k_{IJ}(h)]$ in $[\overline{M}^k_{IJ}]$ is

$$[\overline{M}^k_{IJ}(h)] = \sum_{h_0} P^k_{I,J}(\{T_{h'}\}_{h'\supset h_0}, t_{h_0})\Big|_{t_{h_0}=0}^{t_{h_0}=T_{h_0}} + P^k_{I,J}(0),$$

where the sum is taken after all $h_0\in\mathcal{P}_{d-k_0}$ with $k_0<k$ and such that $h_0\supset\cup_{h'\in I}h'$ and $I\cup J\cup\{h_0\}$ is allowable.

Proof. The map $f_{IJh}:\overline{M}^{k+1}_{IJh}\to\overline{M}^k_{Jh}$ factors through $\mathbb{P}_{h;IJ}\to\overline{M}^k_{Jh}$, where $\mathbb{P}_{h;IJ}$ is the weighted projective fibration of the normal bundle $\mathcal{N}_{\overline{M}^k_{Jh}|\overline{M}^k_{IJ}}$.

It was remarked in section 2 that $\mathbb{P}_{h;IJ}\cong\overline{M}_I^{1,|h|}\times_{\mathbb{P}^n}\overline{M}^k_{Jh}$, where $\overline{M}_I^{1,|h|}$ is the space of 1–stable, $I-$ type, degree $|h|$ rational maps, and the normalization of a closed stratum in $\overline{M}^{1,|h|}:=\overline{M}_{0,1}(\mathbb{P}^n,|h|,1)$. Its associated polynomial

$$(3.9)\qquad P_{\overline{M}_I^{1,|h|}|\overline{M}^{1,|h|}} = \psi_h^{|I|-1}\prod_{i=1}^{|h|-l_I}(H_h+i\psi_h)^{n+1},$$

where H_h is the pull–back of the hyperplane section in $\mathbb{P}^n$ and ψ_h is the cotangent class.

On the other hand, there is a natural birational morphism

$$\overline{M}_{0,1+|Jh|}(\mathbb{P}^n, d-|h\cup(\cup_{h'\in J}h')|)\to\overline{M}^k_{Jh}$$

making $\overline{M}^k_{Jh}$ into an intermediate moduli space of $\overline{M}_{0,1+|Jh|}(\mathbb{P}^n, d - |h \cup (\cup_{h' \in J} h')|)$. The pullback H_h of the hyperplane class in $\mathbb{P}^n$ by the evaluation morphism e_h: $\overline{M}^k_{Jh} \to \mathbb{P}^n$ at the point marked by h is given by the following expression

$$H_h := H + (d - |h \cup (\cup_{h' \in J} h')|)\psi + \sum_{h \subset h''} |h'' \setminus (h \cup (\cup_{h' \in J} h'))| T_{h''},$$

conform Theorem 1 in [LP]). This class is clearly pulled–back from $\overline{M}^k_{Jh}$ and its pullback to $\mathbb{P}_{h;IJ}$ is the class denoted also by H_h.

The map $\overline{M}^{k+1}_{hJ} \to \overline{M}^{k+1}_{IJ}$ pulls back to the gluing map f:

$$\begin{array}{c} \overline{M}_{0,1+|Jh|}(\mathbb{P}^n, d - |h \cup (\cup_{h' \in J} h')|) \times_{\mathbb{P}^n} \overline{M}_{0,1+|I|}(\mathbb{P}^n, |h \setminus (\cup_{h' \in I} h')|) \\ {\scriptstyle f}\Big\downarrow \\ \overline{M}_{0,1+|IJ|}(\mathbb{P}^n, d - |\cup_{h' \in I \cup J} h'|) \end{array}.$$

Let π_1, π_2 be the projections from the domain of f onto $\overline{M}_{0,1+|Jh|}(\mathbb{P}^n, d - |h \cup (\cup_{h' \in J} h')|)$ and $\overline{M}_{0,1+|I|}(\mathbb{P}^n, |h \setminus (\cup_{h' \in I} h')|)$. Let ψ_i be the cotangent line class coming from the i–th marked point. The following relation is also part of Theorem 1 in [LP]:

$$-\pi_1^* \psi_h - \pi_1^* \psi_1 = \sum_{h \subset h'} f^* T_{h'}.$$

The following are also known to be true:

$$\pi_2^* \psi_1 + \pi_1^* \psi_h = f^* T_h \text{ and } \pi_1^* \psi_1 = f^* \psi_1.$$

Putting these together:

$$\pi_2^* \psi_1 = f^* \psi_1 + \sum_{h' \supseteq h} f^* T_{h'}.$$

This is clearly pullback of the class ψ_h from $\mathbb{P}_{h;IJ}$. Thus H_h and ψ_h have been written as linear combinations of H, ψ and boundary divisors. This, together with formula (3.9), determine the polynomial $P_{\overline{M}^k_{hJ} | \overline{M}^k_{IJ}}(T_h)$. The term H_h^{n+1} was introduced in expression (3.7) for division purposes in the case when $I = \emptyset$. It is clearly zero since it comes form a null class in $\mathbb{P}^n$.

The second part follows after applying Lemma 3.20 through successive blow–up steps.

□

The next proposition gives a preliminary formula for $B^*(\overline{M}_{0,1}(\mathbb{P}^n, d)) := B(\overline{M}^d, d)$:

Proposition 3.22. *The $\mathbb{Q}$–algebra $B^*(\overline{M}_{0,1}(\mathbb{P}^n, d))$ is generated over $\mathbb{Q}$ by the divisor classes H, ψ and $\{T_h\}_h$ for all strict subsets h of $M = \{1, ..., d\}$. Its ideal of relations is generated by:*

(1) H^{n+1};

(2) $T_h T_{h'}$ *for all* h, h' *such that* $\{h, h'\}$ *is not allowable, i.e. if* $h \bigcap h' \neq \emptyset, h$ *or* h'.
(3) $\prod_{h \in I} T_h \psi^{|I|-1} \prod_{i=1}^{d-l_I} (H + i\psi)^{n+1}$ *for any partial partition* I *of* M;
(4) $\prod_{h \in I \cup J} T_h \left(\sum_{h_0 \in \mathcal{P}_{\geq d-k}(I,J)} P^k_{I,J}(\{T_h\}_{h \supset h_0}, t_{h_0}) \Big|_{t_{h_0}=0}^{t_{h_0}=T_{h_0}} + P^k_{I,J}(0) \right)$, *for any partial partitions* I *and* J *of* M *such that* $I \bigcup J$ *is still a partial partition. Here* $\mathcal{P}_{\geq d-k}(I, J)$ *is the set of all subsets* h_0 *of* M *with* $|h_0| \geq d - k$ *and such that* $h_0 \supseteq \bigcup_{h \in I} h$ *and* $h_0 \bigcap h = \emptyset$ *whenever* $|h_0| = d - k$ *and* $h \in J$.

Proof. The ring $B^*(\overline{M}_{0,1}(\mathbb{P}^n, d))$ is constructed inductively. Let K be any partial partition of M, and let $k \leq min_{h \in K}\{d - |h|\}$. By successive applications of Lemmas 3.15 and 3.19 we know that

$$B^*(\overline{M}^k_K, k) = \mathbb{Q}[H, \psi, \{T_h\}_{|h| \geq d-k}]/\mathcal{I}_K$$

for some ideal $\mathcal{I}_K$. In particular, we may write

$$B^*(\overline{M}_{0,1}(\mathbb{P}^n, d)) = \mathbb{Q}[H, \psi, \{T_h\}_h]/\mathcal{I}$$

where h ranges over all strict subsets of M. From relation (2) in Lemma 3.19 we conclude that for any element $\alpha \in \mathcal{I}$, there exists a maximal set K and number k such that $\alpha \in \prod_{h \in K} T_h \mathcal{I}_K$. When $K = \emptyset$ and $k = 1$ we obtain elements H^{n+1} and $\psi^{-1} \prod_{i=1}^{d} (H + i\psi)^{n+1}$. Elements (3) in $\mathcal{I}$ arise by Lemma 3.4 when $K \neq \emptyset$ and $k = 1$. Elements (2) in $\mathcal{I}$ are natural compatibility conditions among divisors. For K and $k > 1$ fixed as above, the blow–up locus of $\overline{M}^k_K$ at step k has a number of distinct components, indexed by the choices of subsets $I \subset K$ such that $\sum_{h \in I} |h| \leq d - k$. For such an I, and J such that $J \setminus (K \backslash I) \subset \mathcal{P}_{>d-k}$, consider $h \in \mathcal{P}_{d-k}$ such that $h \supseteq \bigcup_{h' \in I} h'$ and $h \bigcap h' = \emptyset$ for all $h' \in J$. Putting together relations (4) of Lemma 3.15 with relation (3) in Lemma 3.19, plus the polynomial calculations of Lemma 3.21, we obtain elements (4) of $\mathcal{I}$.

□

As is the case with successive blow–ups, many of the relations coming from intermediate steps are superfluous. The next theorem presents a simplified version of the $B^*(\overline{M}_{0,1}(\mathbb{P}^n, d))$ ring structure. The simplification calculations form the Appendix.

Theorem 3.23. *Consider the* $\mathbb{Q}$*–algebra* $B^*(\overline{M}_{0,1}(\mathbb{P}^n, d); \mathbb{Q})$ *generated by the divisor classes* H, ψ *and* $\{T_h\}_h$ *for all proper subsets* h *of* $M = \{1, ..., d\}$, *modulo the ideal generated by:*

(1) H^{n+1};
(2) $T_h T_{h'}$ *for all* h, h' *such that* $h \bigcap h' \neq \emptyset, h$ *or* h';
(3) $T_h T_{h'}(\psi + \sum_{h'' \supseteq h \cup h'} T_{h''})$ *for all* h, h' *nonempty and disjoint;*

(4) $T_h(\sum_{h'\not\subseteq h} P^{d-1}_{\emptyset,h}(\{T_{h''}\}_{h''\supset h'}, t_{h'})\,|^{t_{h'}=T_{h'}}_{t_{h'}=0} + P^{d-1}_{\emptyset,h}(0))$ *for all* $h \subset M$, *where* $T_\emptyset := 1$ *and*

$$P^{d-1}_{\emptyset,h}(\{t_{h''}\}_{h''\supseteq h'}) := (\psi + \textstyle\sum_{h''\supset h'} t_{h''})^{-1}[(H + d\psi + \sum_{h''\supseteq h'} |h''|t_{h''})^{n+1} -$$
$$-(H + (d - |h \cup h'|)\psi + \textstyle\sum_{h''\supset h'}(|h'' \setminus (h \cup h')|)t_{h''})^{n+1})]$$

The group of symmetries S_d *has a natural action on the set of proper subsets of* M.

$A^*(\overline{M}_{0,1}(\mathbb{P}^n, d); \mathbb{Q})$ *is the ring of invariants of the ring* $B^*(\overline{M}_{0,1}(\mathbb{P}^n, d); \mathbb{Q})$ *under the induced action.*

Example 3.24. Here we describe in detail $\overline{M}_{0,1}(\mathbb{P}^n, 2)$ and its Chow ring. In this case $M = \{1, 2\}$. There is only one intermediate space $\overline{M}_{0,1}(\mathbb{P}^n, 2, 1)$, with a blow–up locus $\overline{M}^1(1) \subset \overline{M}_{0,1}(\mathbb{P}^n, 2, 1)$ and two copies of its normalization, $\overline{M}^1_{\{1\}}$ and $\overline{M}^1_{\{2\}}$.

In accord with our considerations in Lemma 3.3, $\overline{M}^1_{\{i\}} \cong \mathbb{P}^n \times \mathbb{P}^n$ for $i \in \{1, 2\}$. Outside its diagonal $\Delta \subset \mathbb{P}^n \times \mathbb{P}^n$, we may think of $\overline{M}^1_{\{i\}}$ as parametrizing lines in $\mathbb{P}^n$ with two marked points q_1 and q_2 on them: q_1 stands for the marked point of $\overline{M}_{0,1}(\mathbb{P}^n, 2)$, q_2 is identified by the set $\{i\}$. The diagonal is isomorphic to $\overline{M}^1_{\{1\},\{2\}}$, parametrizing rational curves with 3 marked points, contracted to a point in $\mathbb{P}^n$. Thus the two projections $\pi_1, \pi_2 : \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$ stand for evaluation maps, the two corresponding pull–backs of the hyperplane class in $\mathbb{P}^n$ are H and $H' = H + \psi$, and the class of $\overline{M}^1_{\{1\},\{2\}} \cong \Delta$ in $\overline{M}^1_{\{i\}} \cong \mathbb{P}^n \times \mathbb{P}^n$ is

$$\sum_{k=0}^{n} H^k H'^{n-k} = \frac{(H + \psi)^{n+1}}{\psi}.$$

On the other hand, by Lemmas 3.3 and 3.21, the image of $\overline{M}^1_{\{i\}}$ in $\overline{M}^1$ has class

$$[\overline{M}^1_{\{i\}}] = \frac{(H + 2\psi)^{n+1}}{\psi},$$

$P_{\overline{M}^1_{\{i\}}|\overline{M}^1}(t)$ is the polynomial

$$P(t) := \frac{(H + 2\psi + t)^{n+1} - (H + \psi)^{n+1}}{\psi + t}.$$

The intersection formula (3.3) on $\overline{M}^1_{\{i\}}$ reads:

$$\frac{(H + 2\psi)^{n+1}}{\psi} = \frac{(H + \psi)^{n+1}}{\psi} + \left.\frac{(H + 2\psi + t)^{n+1} - (H + \psi)^{n+1}}{\psi + t}\right|_{t=0}.$$

The exceptional divisor in $\overline{M}^2$ is $\overline{M}^2_{\{i\}} \cong Bl_\Delta(\mathbb{P}^n \times \mathbb{P}^n) \times_{\mathbb{P}^n \times \mathbb{P}^n} \mathbb{P}(\mathcal{N}_{\overline{M}^1_{\{i\}}|\overline{M}^1})$. The fiber of $\overline{M}^2_{\{i\}} \to \overline{M}^1_{\{i\}}$ over $\{l, q_1, q_2\}$ parametrizes all choices of lines passing through the second marked point q_2 of our line l, whereas the fiber in

$Bl_\Delta \mathbb{P}^n \times \mathbb{P}^n$ over a point in Δ parametrizes choices of lines passing through the third marked point of the curve (since $Bl_\Delta \mathbb{P}^n \times \mathbb{P}^n \cong$ Flag $(1,2,n+1)$).

The following classes are null in the ring $B^*(\overline{M}_{0,1}(\mathbb{P}^n,2);\mathbb{Q})$:

$$H^{n+1} \ , \ \frac{(H+\psi)^{n+1}(H+2\psi)^{n+1}}{\psi} \ , \ T_1T_2\psi \ , \ T_i(H+\psi)^{n+1},$$

$$T_iP(T_i) \ , \ \sum_{i=1}^{2} P(t)\Big|_{t=0}^{t=T_i} + \frac{(H+2\psi)^{n+1}}{\psi}.$$

It is easy to see that the second and fourth classes above are in the ideal generated by the others.

Let $S := T_1 + T_2$, $P := T_1T_2$ and $s_{k+1} := T_1^k + T_2^k$. From the recurrence relation $s_{k+1} = s_kS - s_{k-1}P$ one deduces

$$s_k = \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^i \begin{pmatrix} k-i \\ i \end{pmatrix} S^{k-2i}P^i. \tag{3.10}$$

After taking invariants with respect to the action of S_2 on $\{T_1, T_2\}$, we find that

$$A^*(\overline{M}_{0,1}(\mathbb{P}^n,2);\mathbb{Q}) = \frac{\mathbb{Q}[H,\psi,S,P]}{\mathcal{J}},$$

where $\mathcal{J}$ is generated by H^{n+1}, $P\psi$, $T_1P(T_1)+T_2P(T_2)$ and $P(T_1)+P(T_2)+2\frac{(H+\psi)^{n+1}}{\psi} - \frac{(H+2\psi)^{n+1}}{\psi}$. Using formula (3.10) and the fact that $P\psi = 0$, we may write the last two expressions in terms of S and P. Indeed,

$$P(T_1)+P(T_2) = P(S) + \sum_{k=1}^{n}\sum_{i=1}^{\lfloor k/2 \rfloor} (-1)^i \begin{pmatrix} k-i \\ i \end{pmatrix} \begin{pmatrix} n+1 \\ k+1 \end{pmatrix} S^{k-2i}P^iH^{n-k},$$

$$T_1P(T_1)+T_2P(T_2) =$$

$$= SP(S) + \sum_{k=1}^{n}\sum_{i=1}^{\lfloor k/2 \rfloor} (-1)^i \begin{pmatrix} k-i \\ i \end{pmatrix} \begin{pmatrix} n+1 \\ k \end{pmatrix} S^{k-2i}P^iH^{n-k+1}.$$

4. Appendix

Remark 4.1. If $I \neq \emptyset$, the element (4) in the Proposition 3.22 may be simplified to

$$\prod_{h\in I\cup J} T_h \cdot P^k_{I,J}(\{T_h\}_{h\in\mathcal{P}_{\geq d-k}(I,J)}). \tag{4.1}$$

Indeed, in this case $T_hT_{h'} = 0$ for all $h, h' \in \mathcal{P}_{\geq d-k}(I,J)$ and so, it becomes a nice exercise to see that both the expression (4) in Proposition 3.22 and expression (4.1) above can be written as finite sums

$$\sum_j P^k_{I,J}(\{T_h\}_{h\in\Delta_{h_j}})$$

where each h_j is an element of $\mathcal{P}_{\geq d-k}(I,J)$ and $\Delta_{h_j} = \{h \in \mathcal{P}_{\geq d-k}(I,J); h \supseteq h_j\}$. By comparing coefficients of monomials in T_{h_j} one checks that the same terms appear in both sums with the same coefficients.

Note that when $I = \{h\}$, relation (4.1) tells that $H_h^{n+1} = 0$ on $\overline{M}^k_{Jh}$, where H_h is defined as in formula (3.7).

Remark 4.2. Let $k < k' < d$ and fix $h, h' \subset \{1, ..., d\}$ such that $|h| = d-k$, $|h'| = d-k'$. Recall that the preimage of $\overline{M}^{k'}(h') \subset \overline{M}^{k'}$ in $\overline{M}^{k'}_h$ has two distinct components $\overline{M}^{k'}_h(h'_{in})$ and $\overline{M}^{k'}_h(h'_{out})$, according to the position of the cardinal $(d-k')$-sets in raport to h: $h'_{in} \subset h$ and $h'_{out} \cap h = \emptyset$. The intersection formula

$$T_h[\overline{M}^{k'}(h')] = [\overline{M}^{k'}_h(h'_{in})] + [\overline{M}^{k'}_h(h'_{out})]$$

is a consequence of Lemma 3.21 and relations (4) in Proposition 3.22. Indeed, by Lemma 3.21 :

$$[\overline{M}^{k'}(h')] = \sum_{h''} P^{k'}_{\emptyset,\emptyset}(\{T_{h'''}\}_{h'''\supset h''}, t_{h''}) \mid^{t_{h''}=T_{h''}}_{t_{h''}=0} + P^{k'}_{\emptyset,\emptyset}(0),$$

$$[\overline{M}^{k'}_h(h'_{out})] = T_h(\sum_{h''\not\subseteq h} P^{k'}_{\emptyset,h}(\{T_{h'''}\}_{h'''\supset h''}, t_{h''}) \mid^{t_{h''}=T_{h''}}_{t_{h''}=0} + P^{k'}_{\emptyset,h}(0)),$$

$$[\overline{M}^{k'}_h(h'_{in})] = T_h(\sum_{h''\subset h} P^{k'}_{\emptyset,\emptyset}(\{T_{h'''}\}_{h'''\supset h''}, t_{h''}) \mid^{t_{h''}=T_{h''}}_{t_{h''}=0} + P^{k'}_{\emptyset,\emptyset}(\{T_{h''}\}_{h''\supseteq h})),$$

where the sums are taken after h'' with $|h''| > d-k'$. The last term $T_h P^{k'}_{\emptyset,\emptyset}(\{T_{h''}\}_{h''\supseteq h}))$ is the pullback of the class $[\overline{M}^k_h(h'_{in})]$, as evaluated by the natural change of variables in formula (3.2):

$$H \to H_h := H + (d-|h|)\psi + \sum_{h''\supset h} |h'' \setminus h| T_{h''},$$

$$\psi \to \psi_h(T_h) := \psi + \sum_{h''\supseteq h} T_{h''},$$

and $d \to |h|$ for $\overline{M}_{0,1}(\mathbb{P}^n, |h|)$.

Whenever $h \cap h'' = \emptyset$, the polynomial $P^{k'}_{\emptyset,h}(\{T_{h'''}\}_{h'''\supset h''}, t_{h''})$ differs from $P^{k'}_{\emptyset,\emptyset}(\{T_{h'''}\}_{h'''\supset h''}, t_{h''})$ by a multiple of $\psi_{h\cup h''} T_{h\cup h''}$, which becomes zero on $T_h T_{h''}$ by relation (4) in Proposition 3.22, written for partitions $(I,J) = (\{h, h''\}, \emptyset)$ at the $(d - |h \cup h''|)$–th intermediate step. Whenever $h'' \supset h$, the difference is summarized by the formula:

$$\sum_{h''\supset h} (P^{k'}_{\emptyset,\emptyset} - P^{k'}_{\emptyset,h})(\{T_{h'''}\}_{h'''\supset h''}, t_{h''}) \mid^{t_{h''}=T_{h''}}_{t_{h''}=0} + \tag{4.2}$$

$$+P^{k'}_{\emptyset,\emptyset}(0) - P^{k'}_{\emptyset,h}(0) = P^{k'}_{\emptyset,\emptyset}(\{T_{h''}\}_{h''\supset h}, t_h) \mid_{t_h=0}$$

derived as in the previous remark.

Lemma 4.3. *Let $I \subset \mathcal{P}$ be a nested set and let $h, h', h'' \in \mathcal{P} \setminus I$ denote elements such that $|h| = d-k$ and $|h'| = |h''| = d-k'$, with $k' < k$. Assume for simplicity that the k'–th and k–th steps of blow–up are performed successively (such that the formulas do not explicitely contain exceptional divisors originated in other blow–up steps). The following expression*

$$\sum_{h'} (P_{\overline{M}^{k'}_{Ih'}|\overline{M}^{k'}_I}(T_{h'}) - P_{\overline{M}^{k'}_{Ih'}|\overline{M}^{k'}_I}(0)) + [\overline{M}^{k'}_I(h)]$$

in $B^*(\overline{M}^{k'}_I)[(T_{h'})_{h'}, (T_h)_h]$ is in the ideal generated by:

(1) $\sum_h P_{\overline{M}^k_{Ih}|\overline{M}^k_I}(\{T_{h'}\}_{h'\supset h}, t_h) \mid^{t_h=T_h}_{t_h=0} + \sum_{h'} P_{\overline{M}^k_{Ih}|\overline{M}^k_I}(t_{h'}) \mid^{t_{h'}=T_{h'}}_{t_{h'}=0} + [\overline{M}^k_I(h)]$;
(2) $T_h(\sum_{h'\supset h}(P_{\overline{M}^{k'}_{Ih'}|\overline{M}^{k'}_{Ih}}(T_{h'}) - P_{\overline{M}^{k'}_{Ih'}|\overline{M}^{k'}_{Ih}}(0)) + P_{\overline{M}^{k'}_{Ih'}|\overline{M}^{k'}_{Ih}}(0))$;
(3) $T_{h'}(\sum_{h; h\cap h'=\emptyset} P_{\overline{M}^k_{Ih}|\overline{M}^k_I}(\{T_{h''}\}_{h''\supset h}, t_h) \mid^{t_h=T_h}_{t_h=0} +$
$+ \sum_{h''\neq h'} P_{\overline{M}^k_{Ih}|\overline{M}^k_I}(t_{h''}) \mid^{t_{h''}=T_{h''}}_{t_{h''}=0} + [\overline{M}^k_{Ih'}(h)])$, *for h such that $h \cap h' = \emptyset$;*
(4) $T_{h'} H^{n+1}_{h'}$

Here $[\overline{M}^k_I(h)] = P_{\overline{M}^k_{Ih}|\overline{M}^k_I}(0)$, where $P_{\overline{M}^k_{Ih}|\overline{M}^k_I}$ is thought of as a function of both t_h and $\{t''_h\}_{h''\supset h}$.

Proof. We note that (1) – (3) = (3′), where (3′) is

$$T_{h'}(\sum_{h; h\subset h'} P_{\overline{M}^k_{Ih}|\overline{M}^k_I}(\{T_{h''}\}_{h''\supset h}, t_h) \mid^{t_h=T_h}_{t_h=0} + P_{\overline{M}^k_{Ih}|\overline{M}^k_I}(T_{h'})),$$

and $P_{\overline{M}^k_{Ih}|\overline{M}^k_I}(T_{h'}) = [\overline{M}^k_{Ih'}(h)]$, for h such that $h \subset h'$.

With this, the lemma follows by a direct computation modulo the relations (1)–(4). The initial observation is that

$$P_{\overline{M}^{k'}_{Ih'}|\overline{M}^{k'}_I}(t_{h'}) \mid^{t_{h'}=T_{h'}}_{t_{h'}=0} = P_{\overline{M}^{k'}_{Ih'}|\overline{M}^{k'}_{Ih}}(t_{h'}) P_{\overline{M}^k_{Ih}|\overline{M}^k_I}(t_{h'}) \mid^{t_{h'}=T_{h'}}_{t_{h'}=0} .$$

($P_{\overline{M}^{k'}_{Ih'}|\overline{M}^{k'}_I}(t_{h'})$ as a polynomial is not $P_{\overline{M}^{k'}_{Ih'}|\overline{M}^{k'}_{Ih}}(t_{h'}) P_{\overline{M}^k_{Ih}|\overline{M}^k_I}(t_{h'})$ in general, due to the correction term $H^{n+1}_{h'}$ in the formula for $P_{\overline{M}^{k'}_{Ih'}|\overline{M}^{k'}_I}(t_{h'})$). The right hand side expression above can be written as

$$P_{\overline{M}^{k'}_{Ih'}|\overline{M}^{k'}_{Ih}}(t_{h'}) \mid^{t_{h'}=T_{h'}}_{t_{h'}=0} P_{\overline{M}^k_{Ih}|\overline{M}^k_I}(T_{h'}) + P_{\overline{M}^{k'}_{Ih'}|\overline{M}^{k'}_{Ih}}(0) P_{\overline{M}^k_{Ih}|\overline{M}^k_I}(t_{h'}) \mid^{t_{h'}=T_{h'}}_{t_{h'}=0}$$

Modulo relation (3′), this is

$$-P_{\overline{M}^{k'}_{Ih'}|\overline{M}^{k'}_{Ih}}(t_{h'}) \mid^{t_{h'}=T_{h'}}_{t_{h'}=0} \sum_{h\subset h'} P_{\overline{M}^k_{Ih}|\overline{M}^k_I}(\{T_{h''}\}_{h''\supset h}, t_h) \mid^{t_h=T_h}_{t_h=0} +$$
$$+ P_{\overline{M}^{k'}_{Ih'}|\overline{M}^{k'}_{Ih}}(0) P_{\overline{M}^k_{Ih}|\overline{M}^k_I}(t_{h'}) \mid^{t_{h'}=T_{h'}}_{t_{h'}=0} .$$

As T_h divides $P_{\overline{M}^k_{Ih}|\overline{M}^k_I}(\{T_{h''}\}_{h''\supset h}, t_h)\mid_{t_h=0}^{t_h=T_h}$, summing the above after all h'–s yields, modulo relation (2):

$$\sum_{h'} P_{\overline{M}^{k'}_{Ih'}|\overline{M}^{k'}_I}(T_{h'})\mid_{t_{h'}=0}^{t_{h'}=T_{h'}} =$$
$$= P_{\overline{M}^{k'}_{Ih'}|\overline{M}^{k'}_{Ih}}(0)\sum_h P_{\overline{M}^k_{Ih}|\overline{M}^k_I}(\{T_{h''}\}_{h''\supset h}, t_h)\mid_{t_h=0}^{t_h=T_h} +$$
$$+P_{\overline{M}^{k'}_{Ih'}|\overline{M}^{k'}_{Ih}}(0)P_{\overline{M}^k_{Ih}|\overline{M}^k_I}(t_{h'})\mid_{t_{h'}=0}^{t_{h'}=T_{h'}},$$

which, modulo relation (1) equals $-P_{\overline{M}^{k'}_{Ih'}|\overline{M}^{k'}_{Ih}}(0)[\overline{M}^{k'}_I(h)] = [\overline{M}^{k'}_I(h')]$. □

Lemma 4.4. *Let I, J, be two partial partitions of $M = \{1, ..., d\}$, such that $I \cup J$ is also a partial partition. With the notations introduced in formula (3.7), the expressions*

$$\prod_{h\in I\cup J} T_h P^{d-l_I-1}_{I,J}(\{T_{h'}\}_{h'\in\mathcal{P}_{\geq l_I+1}(I,J)}),\ \prod_{h\in I} T_h(\psi + \sum_{h'\supseteq\bigcup_{h\in I} h} T_{h'})^{|I|-1}$$

(if $|I| > 1$) and $\prod_{h\in I} T_h\psi^{|I|-1}\prod_{i=1}^{d-l_I}(H + i\psi)^{n+1}$ are in the ideal generated by

1. H^{n+1};
2. $T_hT_{h'}$ *if* $h \cap h' \neq \emptyset, h$ *or* h';
3. $T_hT_{h'}(\psi + \sum_{h''\supseteq h\cup h'} T_{h''})$; *for any* $h, h' \neq \emptyset$ *such that* $h \cap h' = \emptyset$;
4. $T_h(\sum_{h'\not\subseteq h} P^{d-1}_{\emptyset,h}(\{T_{h''}\}_{h''\supset h'}, t_{h'})\mid_{t_{h'}=0}^{t_{h'}=T_{h'}} + P^{d-1}_{\emptyset,h}(0))$.

Proof. The first claim can be reduced to the case when $|I| = 1$ and $J = \emptyset$. Indeed, let

$$\psi_{I,J} := \psi + \sum_{h'\in\mathcal{P}_{\geq l_I+1}(I,J)} T_{h'},$$

$$H_{I,J} := H + (d - \sum_{h\in I\cup J}|h|)\psi + \sum_{h'\in\mathcal{P}_{\geq l_I+1}(I,J)} |h'\setminus \bigcup_{h\in I\cup J} h|T_{h'}.$$

Then $P^{d-l_I-1}_{I,J}(\{T_{h'}\}_{h'\in\mathcal{P}_{\geq l_I+1}(I,J)}) = \psi^{|I|-1}_{I,J}H^{n+1}_{I,J}$. Assuming that $|I| \geq 2$, we can choose two distinct $h_1, h_2 \in I$. Modulo relation (3),

$$T_{h_1}T_{h_2}\psi_{I,J} = -T_{h_1}T_{h_2}\sum_{h'\supset h_1\cup h_2, h'\notin\mathcal{P}_{\geq l_I+1}(I,J)} T_{h'}.$$

Furthermore, for each h' as above such that $\cup_{h\in I}h \not\subseteq h'$, we can choose $h_3 \in I$ disjoint from h' and, modulo relations (3) and (2), obtain $T_{h'}T_{h_3}\psi_{I,J} = -T_{h'}T_{h_3}\sum_{h''\supset h'\cup h_3, h''\notin\mathcal{P}_{\geq l_I+1}(I,J)} T_{h''}$. Continuing the same way, we come to the case when $h' \supseteq \cup_{h\in I}h$. Since $h' \notin \mathcal{P}_{\geq l_I+1}(I,J)$, it may be that either $h' = \cup_{h\in I}h$ or $h' = \cup_{h\in I}h \cup h''$, where $\{h \in J; h \subseteq h''\} \neq \emptyset$, or otherwise $\prod_{h\in J} T_hT_{h''} = 0$. Let $I' = \{h'\}$ and let $J' = J$ in the first case, $J' = J \setminus \{h; h \subseteq h''\}$ in the second case. In both cases, $T_{h'}H_{I,J} = T_{h'}H_{I',J'}$ and

moreover, this also equals $T_{h'}H_{I',\emptyset}$ after adjusting by $T_{h'}T_{j'}(\psi+\sum_{h\supseteq h',j'}T_h)$ for every $j'\in J'$.

By the same method, expression $\prod_{h\in I}T_h(\psi+\sum_{h'\supseteq\cup_{h\in I}h}T_{h'})^{|I|-1}$ is reduced to relation (3).

Consider now $I=\{h\}$, $J=\emptyset$. Let $\psi_h(t_h):=\sum_{h'\supset h}T_{h'}+t_h$. We proceed by decreasing induction on $|h|$. Noting that in all our expressions, ψ plays the role of T_h with $h=\{1,...,d\}$, we can start the induction from the relation $H^{n+1}=0$. We can now work out the general induction step. Multiplication of expression (4) by $\psi_h(0)$ gives, modulo expressions (2) and (3):

$$T_h(\textstyle\sum_{h'\cap h=\emptyset}P^{d-1}_{\emptyset,h}(\{T_{h''}\}_{h''\supset h'},t_{h'})\mid^{t_{h'}=T_{h'}}_{t_{h'}=0}(\sum_{h''\supset h,h''\cap h'=\emptyset}T_{h''})+$$
$$+\textstyle\sum_{h'\supset h}P^{d-1}_{\emptyset,h}(\{T_{h''}\}_{h''\supset h'},t_{h'})\mid^{t_{h'}=T_{h'}}_{t_{h'}=0}(\sum_{h'\supset h''\supset h}T_{h''}+\psi_{h'}(T_{h'}))+$$
$$+P^{d-1}_{\emptyset,h}(0)\psi_h(0)),$$

which, after regrouping of terms becomes

$$T_h(\textstyle\sum_{h''\supset h}T_{h''}\sum_{h'\not\supseteq h''}P^{d-1}_{\emptyset,h}(\{T_{h'''}\}_{h'''\supset h'},t_{h'})\mid^{t_{h'}=T_{h'}}_{t_{h'}=0}+ \tag{4.3}$$

$$+\textstyle\sum_{h'\supset h}\psi_{h'}(T_{h'})P^{d-1}_{\emptyset,h}(\{T_{h''}\}_{h''\supset h'},t_{h'})\mid^{t_{h'}=T_{h'}}_{t_{h'}=0}+P^{d-1}_{\emptyset,h}(0)\psi_h(0)). \tag{4.4}$$

As remarked earlier, if $h'\cap h''=\emptyset$, then

$$P^{d-1}_{\emptyset,h}(\{T_{h'''}\}_{h'''\supset h'},t_{h'})\mid^{t_{h'}=T_{h'}}_{t_{h'}=0}=P^{d-1}_{\emptyset,h''}(\{T_{h'''}\}_{h'''\supset h'},t_{h'})\mid^{t_{h'}=T_{h'}}_{t_{h'}=0}$$

and by formula (4.2)

$$\textstyle\sum_{h'\supset h''}(P^{d-1}_{\emptyset,h}-P^{d-1}_{\emptyset,h''})(\{T_{h'''}\}_{h'''\supset h'},t_{h'})\mid^{t_{h'}=T_{h'}}_{t_{h'}=0}+P^{d-1}_{\emptyset,h}(0)-P^{d-1}_{\emptyset,h''}(0)=$$
$$=P^{d-1}_{\emptyset,h}(\{T_{h'''}\}_{h'''\supset h'},t_{h''})\mid_{t_{h''}=0}$$

modulo (2) and (3) and thus, modulo expression (4) written for h'', the summand (4.3) above is

$$-T_h\sum_{h''\supset h}T_{h''}(P^{d-1}_{\emptyset,h}(0)+P^{d-1}_{\emptyset,h}(\{T_{h'''}\}_{h'''\supset h'},t_{h''})\mid_{t_{h''}=0}).$$

On the other hand, summand (4.4) may be written as

$$T_h(\textstyle\sum_{h'\supset h}\psi_{h'}P^{d-1}_{\emptyset,h}(\{T_{h''}\}_{h''\supset h'},t_{h'})\mid^{t_{h'}=T_{h'}}_{t_{h'}=0}+$$
$$+\textstyle\sum_{h'\supset h}\psi_{h'}P^{d-1}_{\emptyset,h}(\{T_{h''}\}_{h''\supset h'},t_{h'})\mid_{t_{h'}=0}(\psi_{h'}(T_{h'})-\psi_{h'}(0))+$$
$$+P^{d-1}_{\emptyset,h}(0)\psi_h(0)).$$

As $\psi_{h'}(T_{h'})-\psi_{h'}(0)=T_{h'}$, putting the two summands together we obtain

$$\sum_{h'\supset h}\psi_{h'}P^{d-1}_{\emptyset,h}(\{T_{h''}\}_{h''\supset h'},t_{h'})\mid_{t_{h'}=0}+P^{d-1}_{\emptyset,h}(0)\psi.$$

In view of definition of $P^{d-1}_{\emptyset,h}$ (formula (3.7)) and of Remark 4.1, the above reduces to $T_hH^{n+1}_{h,\emptyset}$. The induction hypothesis was necessary when applying Remark 4.1. This ends the proof for the second expression in the lemma.

Dependence of the last expression on terms (1)–(4) can be deduced by decreasing induction on I:

$$r(I) := \prod_{h\in I} T_h \psi^{|I|-1} \prod_{i=1}^{d-l_I} (H+i\psi)^{n+1} = \prod_{h\in I} T_h P_{\overline{M}^1_I|\overline{M}^1}(\psi) =$$
$$= \prod_{h\in I\cup\{h'\}} T_h [\overline{M}^1_{Ih'}] P_{\overline{M}^1_{Ih'}|\overline{M}^1}(\psi),$$

for any h' such that $h'\cap(\cup_{h\in I} h) = \emptyset$, where $[\overline{M}^1_{Ih'}]$ is the class of $\overline{M}^1_{Ih'}$ in $\overline{M}^1_I$. Thus by relation (4) of Proposition 3.22, $r(I)$ is in the ideal generated by $r(Ih')$, h' as above. The last induction step coincides with the second expression in the lemma when I is a complete partition of M.

□

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Department of Mathematics, University of British Columbia, Vancouver, B.C., Canada

E-mail address: amustata@math.ubc.ca, dmustata@math.ubc.ca